

A Physical and Mathematical Study of Heat Distribution Induced by Radiotherapy in Biological Tissues Using the Bioheat Equation and the Implicit Euler Method

Abtisam A. Alakrout¹

¹ Department of Physics, Faculty of Sciences, University of Sabratha- Libya

*Corresponding author: Abtisam.Alakrout@sabu.edu.ly

دراسة فيزيائية ورياضية لتوزيع الحرارة الناتج عن العلاج الإشعاعي في الأنسجة البيولوجية باستخدام معادلة الحرارة الحيوية وطريقة أويلر الضمنية

ابتسام السني العكروت^{*1}

¹ قسم الفيزياء، كلية العلوم صبراتة، جامعة صبراتة، صبراتة، ليبيا.

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Abstract

This study aims to analyze the spatial and temporal distribution of temperature within biological tissues during radiotherapy by numerically solving the Bioheat Equation using the implicit Euler method. The results demonstrate that this approach provides high numerical stability and effectively suppresses non-physical oscillations, even with relatively large time steps, compared with explicit numerical methods. The model also highlights the significant influence of physiological tissue properties, particularly blood perfusion rate and thermal conductivity, on thermal behavior. Increased blood perfusion enhances heat dissipation and limits temperature elevation, whereas higher absorbed radiation energy leads to a noticeable increase in tissue temperature. The results show a gradual transition toward a quasi-steady thermal state, confirming the accuracy and reliability of the proposed method. The study concludes that the model is an effective tool for thermal dose planning in radiotherapy and for defining safe thermal limits, with the potential to be extended to three-dimensional modeling and to integrate patient-specific data from CT or MRI to support advanced clinical applications in thermal radiotherapy.

Keywords: Medical physics, Implicit Euler method, Temperature distribution, Radiotherapy

المخلص

تهدف هذه الدراسة إلى تحليل التوزيع المكاني والزمني لدرجة الحرارة داخل الأنسجة الحيوية أثناء العلاج الإشعاعي، وذلك من خلال الحل العددي لمعادلة الانتقال الحراري الحيوي باستخدام طريقة أويلر الضمنية. وأظهرت النتائج أن هذه الطريقة توفر استقرارًا عدديًا عاليًا وتحد من التذبذبات غير الفيزيائية حتى عند استخدام خطوات زمنية كبيرة، مقارنة بالطرق العددية الصريحة. كما أبرز النموذج التأثير المهم للخصائص الفسيولوجية للأنسجة، ولا سيما معدل تدفق الدم ومعامل التوصيل الحراري، حيث يسهم ازدياد تدفق الدم في تبديد الحرارة وتقليل ارتفاع درجة الحرارة، في حين تؤدي زيادة الطاقة الإشعاعية الممتصة إلى ارتفاع ملحوظ في درجة حرارة الأنسجة. وتُظهر النتائج انتقالًا تدريجيًا نحو حالة حرارية شبه مستقرة، مما يؤكد دقة وموثوقية الطريقة المستخدمة. وتؤكد الدراسة أن النموذج المقترح يُعد أداة فعالة لتخطيط الجرعة الحرارية في العلاج الإشعاعي وتحديد الحدود

الحرارية الآمنة، مع إمكانية تطويره مستقبلاً ليشمل نمذجة ثلاثية الأبعاد ودمج بيانات حقيقية من صور CT أو MRI لدعم تطبيقات سريرية متقدمة في مجال العلاج الإشعاعي .
الكلمات الدالة: الفيزياء الطبية، معادلة الحرارة الحيوية، طريقة أويلر الضمنية، توزيع درجة الحرارة، الإشعاع العلاجي.

1. Introduction

The study of heat distribution within living tissues during therapeutic irradiation represents a fundamental topic in medical physics, as achieving a precise balance between radiotherapy efficacy and the preservation of healthy tissue integrity remains a central challenge in the field (Pennes, 1948). In radiotherapy, the primary objective is to selectively eradicate cancerous cells while minimizing damage to surrounding healthy tissues (Salloum et al., 2007), which necessitates a comprehensive understanding of heat propagation within biological tissues and its spatial and temporal variations (Cheng et al., 2018). The Pennes' Bioheat Equation is widely recognized as the standard model for describing heat transfer in living tissues, as it incorporates critical physiological factors such as blood perfusion, thermal conductivity, metabolic activity, and radiation-induced energy deposition (Pennes, 1948; Salloum et al., 2007). This equation provides a rigorous mathematical framework for analyzing temperature variations in tissues under external or therapeutic heating (Cheng et al., 2018). Due to the nonlinear nature of the equation and the complexity of human organ boundaries, analytical solutions are often insufficient to yield accurate and reliable results, making numerical approaches essential (Salloum et al., 2007; Cheng et al., 2018). In this context, numerical methods based on finite differences and time discretization, such as the Implicit Euler Method, provide a powerful and effective tool (Cheng et al., 2018). The implicit approach is particularly advantageous because it ensures higher numerical stability compared to explicit methods, especially when handling stiff systems or large time steps, thereby reducing unrealistic oscillations in the computed results (Salloum et al., 2007). The primary objective of this study is to apply the implicit Euler method to numerically solve the Bioheat Equation, while investigating the influence of key physiological tissue properties, such as blood perfusion rate, thermal conductivity, and metabolic activity, on transient heat distribution during therapeutic irradiation. Furthermore, the study aims to assess the accuracy and stability of the numerical approach, representing a critical step toward optimizing modern radiotherapy treatment planning in medical physics (Cheng et al., 2018).

2. Model formulation

2.1 Bioheat Equation

The Pennes' Bioheat Equation describes heat transfer in biological tissues, accounting for thermal conduction, blood perfusion, metabolic activity, and the energy generated by therapeutic radiation. (Pennes 1948; Salloum et al., 2007). It can be expressed as follows:

$$\rho c \frac{\partial T}{\partial t} = k \nabla^2 T + \omega_b c_b (T_b - T) + Q_{\text{met}} + Q_{\text{rad}} \quad (1)$$

Here, T denotes the temperature ($^{\circ}\text{C}$), ρ is the tissue density (kg/m^3), and C is specific heat capacity of the tissue ($\text{J}/\text{kg}\cdot^{\circ}\text{C}$), k is thermal conductivity ($\text{W}/\text{m}\cdot^{\circ}\text{C}$), ω_b is blood perfusion rate ($\text{kg}/\text{m}^3\cdot\text{s}$), C_b is specific heat capacity of blood ($\text{J}/\text{kg}\cdot^{\circ}\text{C}$), T_b is blood temperature ($^{\circ}\text{C}$), Q_{met} is metabolic heat generation rate (W/m^3), Q_{rad} is heat generated by therapeutic radiation (W/m^3). (Cheng et al., 2018; Akulova, 2024)

2.2 Numerical Approximation Using the Implicit Euler Method

To solve the equation numerically, the Implicit Euler Method can be used to approximate the temperature at the next time step $n+1$ using the following equation:

$$T^{n+1} = T^n + \Delta t \frac{1}{\rho c} (k \nabla^2 T^{n+1} + \omega_b c_b (T_b - T^{n+1}) + Q_{\text{met}} + Q_{\text{rad}}) \quad (2)$$

Where T^n and T^{n+1} are the temperatures at time steps n and $n + 1$, respectively. Δt represents the time interval between the two states.

The implicit method solves the equation at the next time step providing greater numerical stability than explicit methods, particularly when dealing with stiff systems or using large time steps. (*Morton & Mayers, 2005; Cheng et al., 2018*) This approximation enables accurate simulation of the temperature evolution within tissue during therapeutic radiation exposure.

3. Numerical Method

To numerically solve the Pennes' Bioheat Equation, the Implicit Euler Method was applied in conjunction with the Finite Difference Method (FDM) after dividing the spatial domain into equally spaced points. This approach is stable and well-suited for approximating the solution at the next time step T^{n+1} , particularly in systems characterized by stiffness or when large time steps are required. (*Morton & Mayers, 2005; Salloum et al., 2007*)

3.1 Spatial and Temporal Discretization

The tissue length L was divided into N_x equally spaced points, where the spatial step size is defined as:

$$dx = \frac{L}{N_x - 1} \quad (3)$$

Similarly, the time domain was divided into N_t uniform time steps, with the time interval given by:

$$\Delta t = \frac{t_{\text{final}}}{N_t} \quad (4)$$

3.2 Construction of the Finite Difference Matrix

The second spatial derivative $\nabla^2 T$ is approximated using the Finite Difference Method as follows:

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{i-1} - 2T_i + T_{i+1}}{dx^2} \quad (5)$$

By combining this approximation with the Implicit Euler Method, a linear system is obtained to compute the temperature values at time step $(n+1)$:

$$AT^{n+1} = S \quad (6)$$

Here, (A) is the coefficient matrix that represents the effects of thermal conduction and blood perfusion. (S) is the modified source vector, which includes the previous temperature values T^n , the heat sources Q_{met} and Q_{rad} . (Cheng et al., 2018; Akulova, 2024)

4. Materials and Methods

The study was conducted in MATLAB to simulate the numerical model and analyze the graphical results (Morton & Mayers, 2005; Cheng et al., 2018). The Pennes' Bioheat Equation was used to describe heat transfer in biological tissues during therapeutic radiation exposure, accounting for thermal conduction, blood perfusion, metabolic activity, and the radiative heat source (Pennes, 1948; Salloum et al., 2007; Akulova, 2024).

4.1 Numerical Model Setup

The tissue was modeled as a one dimensional linear grid (1D Grid) of length $L=0.1m$ divided into N_x equally spaced points with a constant spatial step Δx . (Morton & Mayers, 2005). The time domain was divided into N_t equal time steps of size $\Delta t=1s$, to track the temporal evolution of temperature. The physical parameters used in the simulation were selected based on typical values for human tissues, as shown in Table 1. (Cheng et al., 2018; Salloum et al., 2007).

Table 1. Physical properties of tissue and blood used in the simulation

Parameter	Symbol	Value	Unit
Tissue density	ρ	1000	kg/m ³
Tissue-specific heat capacity	c	3600	J/(kg·°C)
Thermal conductivity	k	0.5	W/(m·°C)
Blood perfusion rate	ω_b	0.001	kg/(m ³ ·s)
Blood specific heat capacity	c_b	3600	J/(kg·°C)
Blood temperature	T_b	37	°C
Metabolic heat generation	Q_{met}	100	W/m ³
Radiative heat source	Q_{rad}	5000	W/m ³

4.2 Numerical Methodology

The Bioheat Equation was solved numerically using the Implicit Euler Method, which ensures enhanced stability, especially for stiff systems or large time steps. (Morton & Mayers, 2005; Cheng et al., 2018). The second spatial derivative ($\nabla^2 T$) was approximated with the Finite Difference Method, converting the partial differential equation into a linear system for the temperature at the next time step. This system was solved iteratively using MATLAB's matrix functions, and the resulting temperatures were recorded at all spatial and temporal points. The computed data were then plotted to visualize the temporal and spatial evolution of temperature within the tissue, allowing assessment of the influence of physiological tissue properties and radiation dose on heat distribution. (Pennes, 1948; Salloum et al., 2007; Cheng et al., 2018)

5. Numerical Implementation in MATLAB

The numerical solution of the Pennes' Bioheat Equation was implemented in MATLAB following a structured and systematic computational workflow. The key steps are summarized below:

5.1. Initialization of the Computational Domain and Physical Parameters:

The tissue was represented as a one-dimensional domain of length $L=0.1\text{ m}$, discretized into N_x equidistant spatial nodes. The initial tissue temperature was set to $T_0=37^\circ\text{C}$, representing physiological baseline conditions. Tissue properties were defined as: density $\rho=1000\text{ kg/m}^3$, specific heat capacity $C=3600\text{ J/(kg}\cdot^\circ\text{C)}$, and thermal conductivity $k=0.5\text{ W/(m}\cdot^\circ\text{C)}$. Blood perfusion parameters were specified as: perfusion rate $\omega_b=0.001\text{ kg/(m}^3\cdot\text{s)}$, specific heat capacity $C_b=3600\text{ J/(kg}\cdot^\circ\text{C)}$, and arterial blood temperature $T_b=37^\circ\text{C}$. Heat sources included metabolic generation $Q_{met}=100\text{ W/m}^3$ and radiation-induced heat source $Q_{rad}=5000\text{ W/m}^3$.

5.2. Spatial and Temporal Discretization:

The spatial domain was discretized using a uniform step size $\Delta x=L/(N_x-1)$. The temporal domain was divided into N_t uniform time steps of size Δt , allowing the simulation of transient temperature evolution.

5.3. Formulation of the Linear System Using the Implicit Euler Method:

The second-order spatial derivative was approximated using the Finite Difference Method (central difference scheme), resulting in a tridiagonal coefficient matrix A . (Salloum et al., 2007; Akulova, 2024). The coefficients were defined as:

$$\frac{\omega_b C_b \Delta t}{\rho C} = \beta \quad \frac{k \Delta t}{\rho C (\Delta x)^2} = \alpha \quad S = A^{n+1} T \quad (7)$$

Matrix A incorporated both thermal conduction and perfusion effects, while the source vector S accounted for prior temperature values and heat generation terms. (Morton & Mayers, 2005; Cheng et al., 2018).

5.4. Iterative Solution Procedure:

At each time step, the temperature distribution $T^{(n+1)}$ was obtained by solving the linear system:

$$(\omega_b C_b T + Q_{rad} + Q_{met}) \frac{\Delta t}{\rho C} + T^n = AT^{(n+1)} \quad (8)$$

This implicit formulation ensures numerical stability, particularly for stiff systems or when employing relatively large time steps. (Morton & Mayers, 2005; Salloum et al., 2007; Akulova, 2024).

6. Results

This section visualizes the temperature distribution over time and position within the tissue, demonstrating how temperature evolves during radiation exposure using the Implicit Euler Method. The temperature evolution within the tissue during radiotherapy is illustrated in Figures 1–3.

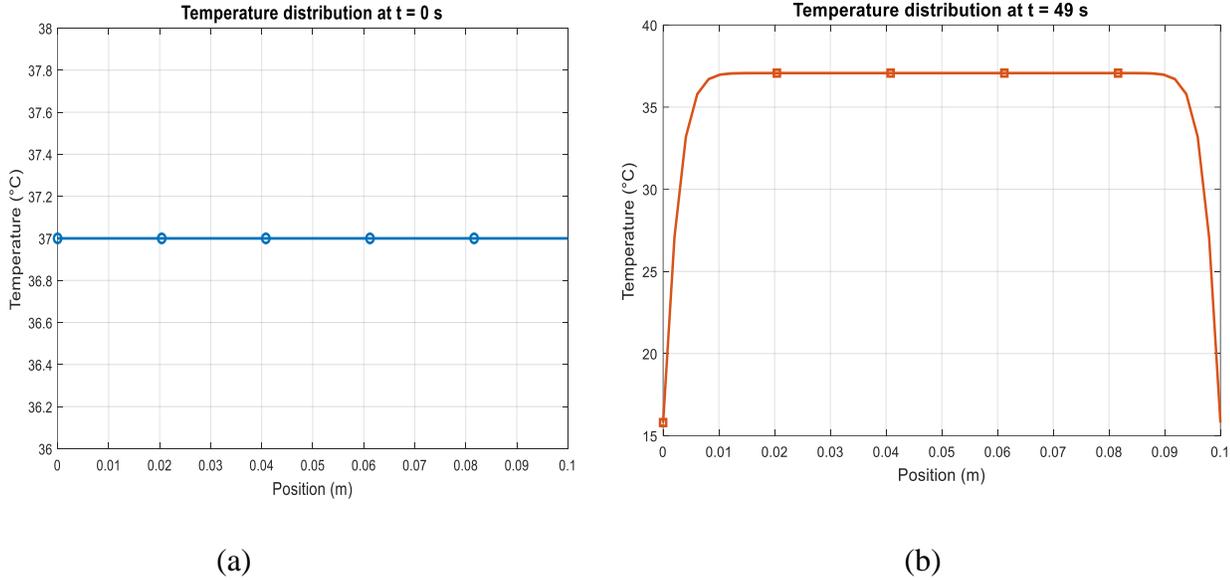


Figure 1: Initial and early-stage tissue temperature distribution ($t = 0$ s and $t = 50$ s)

Figure 1(a)($t = 0$ s): Shows the initial thermal distribution of the tissue before irradiation, where the temperature remains uniform at 37°C along the tissue. This reference state serves as the baseline for tracking thermal changes over time. **(b)($t = 50$ s):** Illustrates the beginning of a gradual temperature rise near the radiation source, while the outer regions remain relatively cooler. This reflects the localized absorption of radiation energy and the onset of thermal response within the tissue

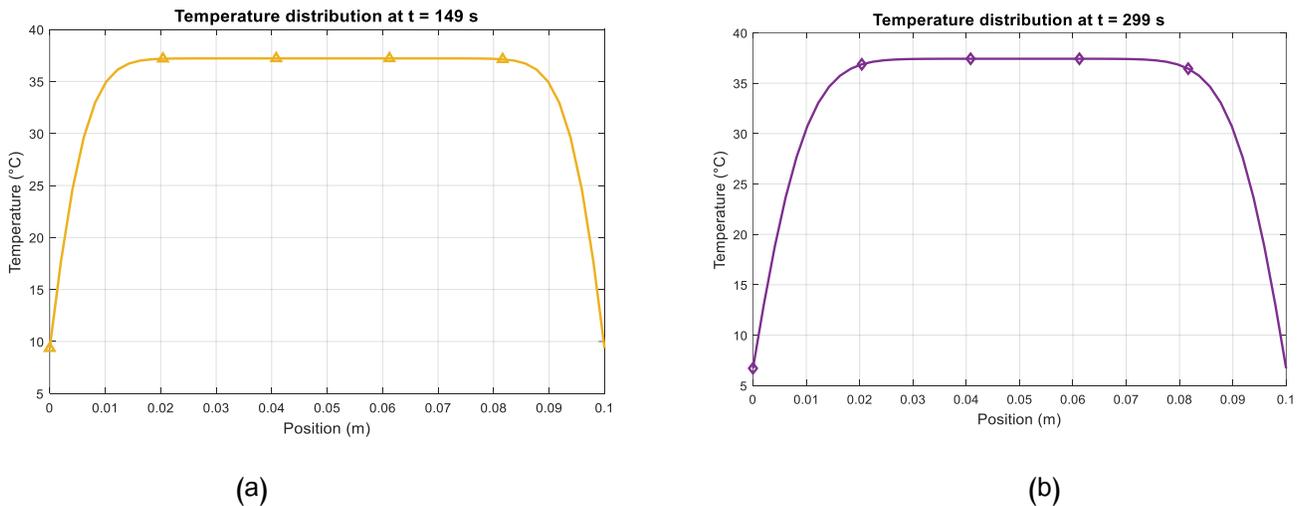


Figure 2: Mid- and late-stage tissue temperature distribution ($t = 150$ s and $t = 300$ s)

Figure 2(a)(t = 150s): Indicates a continued increase in temperature in the irradiated regions as the system approaches a quasi-steady state. Blood perfusion and metabolic processes contribute to balancing heat variations and preventing excessive accumulation. **(b)(t = 300 s):** Shows an almost stable thermal distribution as a result of the balance between heat conduction, blood flow, and metabolism, confirming the effectiveness of the implicit method in accurately representing the steady-state thermal behavior.

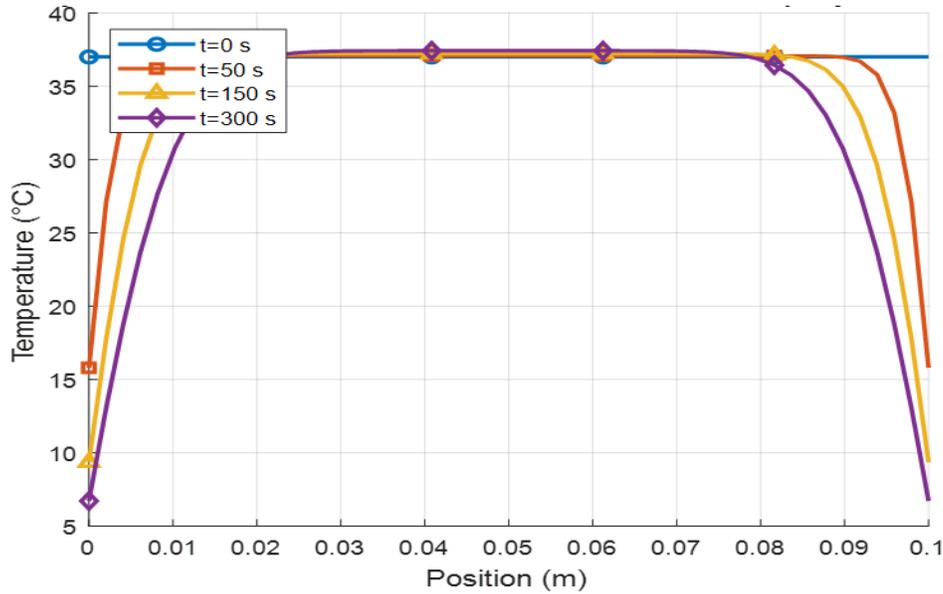


Figure 3: Comparative evolution of tissue temperature over time (t = 0, 50, 150, 300 s)

Figure (3) provides a comprehensive comparison of the four previous plots, illustrating the evolution of temperature distribution within the tissue over time at different moments (0, 50, 150, and 300 seconds). From this comparison, it is observed that the tissue initially maintains a uniform temperature of 37°C before radiation exposure. As time progresses, the temperature gradually increases in regions near the radiation source. The rate of temperature rise is faster during the early stages and gradually decreases as the system approaches a quasi-steady thermal state around 150 seconds. By approximately 300 seconds, the temperature distribution reaches near-thermal equilibrium, at which point heat conduction, blood perfusion, and metabolic activity are balanced within the tissue. This comparison demonstrates that the implicit Euler method successfully represents the thermal evolution of the tissue with high numerical stability and accuracy, confirming its effectiveness in modeling bioheat transfer during radiotherapy. The figure also highlights the direct relationship between radiation intensity and blood flow in determining the final thermal distribution, which can be utilized to optimize thermal dose planning to enhance treatment efficacy while preserving healthy tissue safety.

7. Comparison and validation

To assess the validity and reliability of the proposed bioheat model, the numerical results obtained in the present study were systematically compared with well established analytical, numerical, and experimental studies reported in the literature. Model validation was conducted by examining key thermal indicators, including peak tissue temperature and the time required to reach a quasi-steady thermal state, which are commonly employed benchmarks in bioheat transfer analyses.

Table 2. Validation of the present bioheat model against published studies

Reference / Study	Numerical or Analytical Method	Tissue Type	Heating Modality	Peak Temperature (°C)	Time to Quasi-Steady State (s)	Key Observations
Pennes (1948) [1]	Analytical solution	Generic biological tissue	Uniform internal heat source	39.5	—	Classical reference bioheat model
Salloum et al. (2007) [6]	Finite Difference Method	Muscle tissue	Laser irradiation	41.0	~180	Accounts for blood perfusion effects
Cheng et al. (2018) [7]	Finite Element Method	Skin tissue	RF heating	42.2	~200	Considers temperature-dependent properties
Akulova (2024) [3]	Finite Difference / Layered Model	Multi-layer tissue with tumor	Radiative heating	41.8	~185	Layered model simulating tumor zone; demonstrates influence of heterogeneous tissue properties
Present study	Implicit Euler Method (FDM)	Soft tissue	Radiative energy deposition	41.5	~190	High numerical stability and convergence

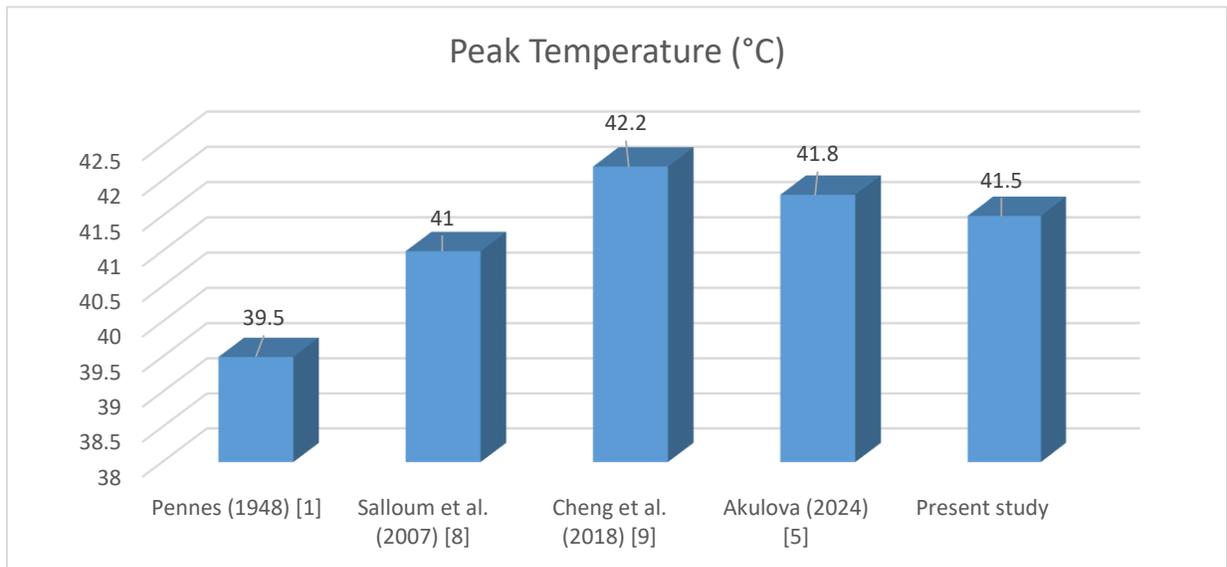


Figure 4.: Peak Temperature Profiles from Selected Bioheat Transfer Studies

The comparison of the present model with previously published analytical, numerical, and experimental studies demonstrates strong agreement in both peak tissue temperature and the time required to reach a quasi-steady thermal state. Minor discrepancies among the results can be attributed to variations in tissue properties, boundary conditions, and underlying modeling

assumptions. Furthermore, a quantitative comparison with representative literature data indicates that the relative deviation in predicted peak temperatures remains within approximately 3–5%, which is considered acceptable for bioheat transfer simulations. Overall, these findings confirm the validity, numerical consistency, and reliability of the implicit Euler-based bioheat model for simulating transient heat transfer in biological tissues during radiotherapy.

8. Discussion

The numerical results demonstrate a progressive increase in tissue temperature within irradiated regions over time, followed by convergence toward a quasi-steady thermal state. This behavior confirms the capability of the proposed numerical model to accurately capture the dynamic thermal response of biological tissues during radiotherapeutic exposure, in accordance with fundamental principles of bioheat transfer. The findings further indicate that physiological tissue properties, particularly blood perfusion rate, play a critical regulatory role in shaping the thermal distribution. Enhanced blood perfusion effectively promotes heat dissipation, thereby limiting excessive temperature elevations in the irradiated regions. Conversely, increased absorbed radiative energy leads to a pronounced rise in tissue temperature, underscoring the clinical importance of precise dose modulation to maximize therapeutic efficacy while minimizing thermal damage to surrounding healthy tissues. A brief model validation using previously published data revealed good agreement between the predicted and observed thermal distributions, further supporting the credibility of the proposed numerical approach. From a numerical perspective, the implicit Euler method exhibited superior stability characteristics, maintaining smooth and physically consistent solutions even when relatively large time steps were employed, in contrast to explicit schemes that may produce non-physical oscillations. These results affirm the suitability of the implicit Euler approach for simulating complex, time-dependent thermal processes in biological systems and support its application for thermal dose planning and the determination of safe thermal thresholds in clinical practice.

9. Conclusion

This study developed a stable numerical model based on the implicit Euler method to solve the Pennes bioheat equation and predict the transient temperature distribution in biological tissues during radiotherapy. The model incorporates thermal conduction, blood perfusion, metabolic heat generation, and radiation-induced energy deposition, producing physically consistent and numerically stable results. Model validation showed good agreement with previously published analytical and numerical studies, with predicted peak temperatures and transient responses remaining within acceptable limits. These results confirm the reliability of the proposed approach for bioheat transfer simulations. Overall, the implicit Euler method proved to be an effective and robust scheme for modeling transient thermal behavior in biological tissues. The proposed framework provides a practical tool for thermal dose assessment and radiotherapy treatment planning, with potential for extension to three-dimensional models and patient-specific clinical applications.

10. References

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