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بعض نظريات النقاط الثابتة المشتركة في الفضاء المتري الموسع b_2 عبر الدوال المقبولة

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Some Common fixed points Theorems in an extended b_2-metric space via Admissible Mapping

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ABSTRACT

This paper examines the existence and the uniqueness of common fixed points for mappings that atisfy an (T, S) and (α, β)-orbital cyclic admissibility condition within the context of extended b_2-metric spaces. The main theorem outlines the conditions necessary for common fixed points of a pair of self-mappings (T, S) under the admissibility criteria, which include a contractive-like condition involving α and β. Ultimately, the paper offers corollaries and illustrations for single-valued mappings, demonstrating how the established theory can be utilized in various contexts.

Keywords: Common Fixed points, Extended b_2-Metric Space, Orbital Cyclic- Admissible Mapping**.**

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ملخـــــــص البحـــــــث

تبحث هذه الورقة في وجود وتفرد النقاط الثابتة المشتركة للدوال الدورية المدارية (T, S) والدوال (α، β) التي تحقق شرط القبول الدوري المداري في سياق الفضاء المتري b_2 الموسع. تحدد النظرية الرئيسية الشروط اللازمة للنقاط الثابتة المشتركة لزوج من الدوال الذاتية (T, S) وفقًا لمعايير القبول، والتي تتضمن شرطًا شبيهًا بالانكماش يشمل α وβ. في النهاية، تقدم الورقة نتائج وتوضيحات للدوال أحادية القيمة ، موضحةً كيفية استخدام النظرية المُعتمدة في سياقات مختلفة.

الكلمات الدالة: النقاط الثابتة المشتركة , الفضاء المتري b_2 الموسع, الدوال الدورية المدارية المقبولة.

1. Introduction

In recent years, numerous researchers have investigated common fixed points of mappings that adhere to various contractive conditions. This field has a wide range of significant applications in applied mathematics and the sciences. In 1976, Jungck [10] established a common fixed point theorem for commuting maps, contingent upon the continuity of at least one of the mappings. In 1982, Sessa , introduced the notion of weak commutativity for pairs of self-maps. He demonstrated that weakly commuting pairs of maps within a metric space are indeed commuting; however, the reverse is not necessarily true. The idea of a b-metric space was first proposed by Czerwik in 1993 and 1998 [4,5], which led to the development of several fixed-point results. Later, Zead Mustafa and colleagues (2014) [12] introduced a generalized metric space known as the b_2-metric space, which encompasses both the 2-metric space and the b-metric space. In 2017, Kamran et al [11], explored an extended b-metric space and derived unique fixed-point results. More recently, in 2018, Elmabrouk and Alkaleeli [6-9] introduced a new type of generalized b_2-metric space, referred to as extended b_2-metric spaces, which generalizes both the b_2-metric space and the extended b-metric space. Subsequently, we validated some fixed-point theorems by Elmabrok and Al-Muqsabi in (2021-2022) [8]. This article specifically examines the existence of common fixed points for a certain class of mappings called (α, β) orbital cyclic-admissible mappings, all within the framework of extended b_2-metric spaces. These spaces, which generalize b-metric spaces and other related structures, create a broader context for addressing fixed point problems. We build on and extend existing fixed point theorems found in the literature. A new concept, (α, β) orbital cyclic admissibility, is introduced for a pair of self-mappings (S, T) within a complete extended b_2-metric space. This new idea combines orbital cyclic mappings with (α, β) admissible mappings, providing a robust framework for studying fixed point existence. A significant theorem (Theorem 3.3) is established to guarantee the existence of a common fixed point for the pair of mappings under specific conditions, including a contractive conditions modulated by the α and β functions, as well as conditions on the θ function associated with the extended b_2-metric. Illustrative examples are provided to demonstrate the applicability of our main result. In addition, we explore the existence of fixed points for a single self-mapping T under the (α, β) orbital cyclic admissibility condition. Corollaries 3.6 and 3.8 present corresponding fixed point theorems for this situation, supported by relevant examples. Lastly, we tackle the uniqueness of the fixed point under an additional condition (A), leading to Theorem 4.1, which asserts that the common fixed point established by Theorem 3.3 is unique when condition (A) is satisfied.

The subsequent sections will provide crucial background information on extended b_2-metric and will confirm various fixed-point theorems. Section 3 will examine notable fixed-point results in the framework of extended b_2-metric space, utilizing (S,T) and (α,β) – orbital-cyclic admissible mappings. This will be accompanied by comprehensive proofs of the primary results and relevant illustrative examples.

2. Preliminaries

Following this groundbreaking finding on the extended b-metric, numerous authors have documented various intriguing outcomes in this realm (see, for instance, [1-3, 6-9, 13] and the pertinent references therein). First and foremost, we will revisit some definitions of different types of generalized metric spaces, along with several theorems and characteristics of extended b_2-metric spaces, which will be utilized later.

Let X be a nonempty set, $T: X \to X$ and $S: X \to X$ are two self-mappings. We say that $x \in X$ is a common fixed point of T and S if $T(x) = x = S(x)$, and CFix(T) denotes the set of common fixed points of T and S.

Definition 2.1 [2] Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that T is an α-orbital admissible if, for all $x, y \in X$, we have

$$\alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1. \qquad (2.1)$$

Definition 2.2 [1] A set X is regular with respect to mapping $\alpha : X \times X \to [0,\infty)$ if, whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_{(k)}}\}$ of $\{x_n\}$ such $that$ $\alpha(x_{n_{(k)}}, x) \geq 1$ and $\alpha(x, x_{n_{(k)}}) \geq 1$ for all n.

Definition 2.3 [2] Suppose that T, S are two self-mappings on a complete extended-*b* metric space (X, d_θ). Suppose also that there are two functions $\alpha, \beta : X \times X \to [0,\infty)$ such that, for any $x \in X$,

$$\alpha(x, Tx) \geq 1 \Rightarrow \beta(Tx, STx) \geq 1,$$

and

$$\beta(x, Sx) \geq 1 \Rightarrow \alpha(Sx, TSx) \geq 1. \tag{2.2}$$

Then we say that the pair S, T is an (α,β)-orbital-cyclic admissible pair.

Lemma 2.4 [2] Let $(\mathrm{X}, \mathrm{d}_\theta)$ be an extended b-metric space. If there exists $\mathrm{q} \in [0,1)$ such that the sequence $\{\mathrm{x_n}\}$ for an arbitrary $\mathrm{x_0} \in \mathrm{X}$ satisfies $\lim_{\mathrm{n,m}\to\infty} \theta(\mathrm{x_n}, \mathrm{x_m}) < \frac{1}{q}$

, and also

$$0 < \mathrm{d}_\theta(\mathrm{x}_{n+1}, \mathrm{x}_m) \leq q\mathrm{d}_\theta(\mathrm{x}_{n-1}, \mathrm{x}_m) \tag{2.3}$$

for any $\mathrm{n} \in \mathrm{N}$, then the sequence $\{\mathrm{x_n}\}$ is Cauchy in X.

Definition 2.5 [13] Let X be a nonempty set, $T : X \to X$, and $\alpha, \beta : X \times X \to [0,\infty)$. We say that T is an (α,β) −orbital-cyclic admissible mapping if

$$\alpha(x, Tx) \geq 1 \text{ implies } \beta(Tx, T^2x) \geq 1$$

and

$$\beta(x, Tx) \geq 1 \text{ implies } \alpha(Tx, T^2x) \geq 1 \tag{2.4}$$

for all $x \in X$.

Definition 2.6 [14] Given a mapping $T: X \to X$ and $x_0 \in X$, for all $n \in \mathbb{N}$, the orbit of x_0 with respect to T is defined as the following sequences of points ,

$$\mathcal{O}(x_0) = \{ x_0, Tx_0, \dots, T^n x_0, \dots \}.$$

The subsequent findings correspond to those of Elmabrouk and Alkaleeli [6-8] within an expanded b_2-metric framework.

Definition 2.7 [6] Let X be a nonempty set and $\theta: X \times X \times X \to [1,\infty)$ be a mapping. A function $d_\theta: X \times X \times X \longrightarrow [0,\infty)$ is an extended b_2-metric on X if for all $a, x, y, z \in X$, the following conditions hold:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d_\theta(x,y,z) \neq 0$,
2) If at least two of three points x, y, z are the same, then $d_\theta(x,y,z) = 0$.
3) $d_\theta(x,y,z) = d_\theta(x,z,y) = d_\theta(y,x,z) = d_\theta(y,z,x) = d_\theta(z,x,y) = d_\theta(z,y,x)$, the symmetry,
4) $d_\theta(x,y,z) \leq \theta(x,y,z)[d_\theta(x,y,a) + d_\theta(y,z,a) + d_\theta(z,x,a)]$, the rectangle inequality.

Then d_θ is called an extended b_2-metric on X and the pair (X, d_θ) is called an extended b_2-metric space.

Remark 2.8. [6]

It is obvious that the class of an extended b_2-metric space is larger than b_2-metric space, because if $\theta(x,y,z) = s$, for $s \geq 1$ then we obtain the definition of a b_2-metric space.

Example 2.9. [8] Let $X = [0,1]$. Define $\theta : X \times X \times X \longrightarrow [1,\infty)$ by

$$\theta(x,y,z) = \frac{1+x+y+z}{x+y+z} \qquad \text{for all } x,y,z \in X.$$

And $d_\theta: X \times X \times X \longrightarrow [0,\infty)$ by

$$d_\theta(x,y,z) = \begin{cases} \dfrac{1}{xyz} & \text{if } x,y,z \in (0,1] \text{ and } x \neq y \neq z, \\ 0 & \text{if } x,y,z \in [0,1] \text{ and at least two of } x, y, \text{ and } z \text{ are equal}, \\ \dfrac{1}{xy} & \text{if } x,y \in (0,1] \text{ and } z = 0. \end{cases}$$

Then (X, d_θ) is an extended b_2-metric space .

Definition 2.10. [9] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an extended b_2-metric space (X, d_θ).

1. A sequence $\{x_n\}$ is a Cauchy sequence if and only if $d_\theta(x_n, x_m, a) \to 0$, when $n, m \to \infty$. for all $a \in X$.
2. A sequence $\{x_n\}$ is convergent to $x \in X$, if for all $a \in X$, there exists $x \in X$, such that $\lim_{n\to\infty} d_\theta(x_n, x, a) = 0$.
3. An extended b_2-metric space (X, d_θ) is called complete if every Cauchy sequence is convergent sequence.

Definition 2.11. [7] Let (X, d_θ) be an extended b_2 −metric space. The extended b_2 − metric d_θ is called continuous if

$$d_\theta(x_n, x, a) \to 0 \ and \ d_\theta(y_n, y, a) \to 0 \implies d_\theta(x_n, y_n, a) \longrightarrow d_\theta(x, y, a),$$

for all sequence $\{x_n\}, \{y_n\}$ in X and $x, y, a \in X$.

Main Results

In this section, we introduce the notion of $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-orbital-cyclic admissible in the setting of extended $\boldsymbol{b_2}$-metric spaces. Then, extend common fixed point theorems for pair of mappings in an extended $\boldsymbol{b_2}$-metric space. Also, some examples in support of our main results are provided.

Definition 3.1

Suppose that T and S are two self-mappings on a complete extended b_2-metric space (X, d_θ). Suppose also that there are two functions $\alpha, \ \beta : X \times X \times X \to [0, \infty)$ such that, for $x, a \in X$,

$$\left.\begin{aligned} \alpha(x, Tx, a) \geq 1 \quad &\Longrightarrow \quad \beta(Tx, STx, a) \geq 1, \\ \beta(x, Sx, a) \geq 1 \quad &\Longrightarrow \quad \alpha(Sx, TSx, a) \geq 1. \end{aligned}\right\} \tag{3.1}$$

Then we say that the pair $S, \ T$ is an $(\alpha, \ \beta)$-orbital-cyclic admissible pair.

Lemma 3.2

Let (X, d_θ) be an extended b_2- metric space. If there exists $q \in [0,1)$ such that the sequence $\{x_n\}$ for an arbitrary $x_0 \in X$ satisfies

$$\lim_{n,m\to\infty} \theta(x_n, x_m, a) < \frac{1}{q},$$

And

$$0 < d_\theta(x_n, x_{n+1}, a) \le q d_\theta(x_{n-1}, x_n, a) \qquad (3.2)$$

for any $n \in \mathbb{N}, a \in X$, then the sequence $\{x_n\}$ is Cauchy in X.

Proof

Let $\{x_n\}_{n\in\mathbb{N}}$ be a given sequence. By employing inequality (3.2) recursively, we derive that

$$0 < d_\theta(x_n, x_{n+1}, a) \le q^n d_\theta(x_0, x_1, a). \qquad (3.3)$$

Since $q \in [0,1)$, we find that

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}, a) \ = 0,$$

On the other hand, by (iii), together with triangular inequality (iv), for $p \ge 1$, we derive that

$$d_\theta(x_n, x_{n+p}, a) \le \theta(x_n, x_{n+p}, a) \begin{bmatrix} d_\theta(x_n, x_{n+p}, x_{n+1}) \\ + \ d_\theta(x_{n+p}, a, x_{n+1}) \\ +d_\theta(a, x_n, x_{n+1}) \end{bmatrix},$$

$$\le \begin{bmatrix} \theta(x_n, x_{n+p}, a) d_\theta(x_n, x_{n+p}, x_{n+1}) \\ +\theta(x_n, x_{n+p}, a) d_\theta(x_{n+p}, a, x_{n+1}) \\ +\theta(x_n, x_{n+p}, a) d_\theta(a, x_n, x_{n+1}) \end{bmatrix},$$

$$\le \begin{bmatrix} \theta(x_n, x_{n+p}, a) q^n d_\theta(x_0, x_1, a) \\ +\theta(x_n, x_{n+p}, a) q^n d_\theta(x_0, x_1, x_{n+p}) \\ +\theta(x_n, x_{n+p}, a) q^n d_\theta(x_{n+p}, a, x_{n+1}) \end{bmatrix},$$

$$\le \begin{bmatrix} \theta(x_n, x_{n+p}, a) q^n d_\theta(x_0, x_1, a) \\ +\theta(x_n, x_{n+1}, a) q^n d_\theta(x_0, x_1, x_{n+p}) \\ +\theta(x_n, x_{n+p}, a)\theta(x_{n+1}, x_{n+p}, a) \begin{bmatrix} d_\theta(x_{n+1}, x_{n+2}, a) \\ +d_\theta(x_{n+1}, x_{n+2}, x_{n+p}) \\ +d_\theta(x_{n+2}, x_{n+p}, a) \end{bmatrix} \end{bmatrix},$$

$$\leq \left[\begin{bmatrix} \theta(x_n, x_{n+p}, a)q^n \\ +\theta(x_n, x_{n+p}, a)\theta(x_{n+1}, x_{n+p}, a)q^{n+1} \end{bmatrix} d_\theta(x_0, x_1, x_{n+p}) \right.$$

$$\left. + \theta(x_n, x_{n+p}, a)\theta(x_{n+1}, x_{n+p}, a)d_\theta(x_{n+2}, x_{n+p}, a)\right],$$

$$\leq \left[\begin{bmatrix} \theta(x_n, x_{n+p}, a)q^n \\ +\theta(x_n, x_{n+p}, a)\theta(x_{n+1}, x_{n+p}, a)q^{n+1} \\ + \cdots + \cdots \\ +\theta(x_n, x_{n+p}, a)\theta(x_{n+1}, x_{n+p}, a)\cdots\cdots \\ \cdots\cdots\theta(x_{n+p-2}, x_{n+p}, a)\theta(x_{n+p-1}, x_{n+p}, a)q^{n+p-1} \end{bmatrix} [d_\theta(x_0, x_1, a) + d_\theta(x_0, x_1, x_{n+p})]\right],$$

$$= d_\theta(x_0, x, a) \sum_{i=1}^{n+p-1} q^i \prod_{j=1}^{i} \theta(x_{n+j}, x_{n+p}, a).$$

Notice the inequality above is dominated by

$$\sum_{i=1}^{n+p-1} q^i \prod_{j=1}^{i} \theta(x_{n+j}, x_{n+p}, a) \leq \sum_{i=1}^{n+p-1} q^i \prod_{j=1}^{i} \theta(x_j, x_{n+p}, a).$$

On the other hand, by employing the ratio test, we conclude that the series $\sum_{i=1}^{\infty} a_i$, where

$$a_i = q^i \prod_{j=1}^{i} \theta(x_j, x_{n+p}, a)$$

Converges to some $S \in (0, \infty)$. Indeed,

$$\lim_{i\to\infty} \frac{a_{i+1}}{a_i} = \lim_{i\to\infty} q\theta(x_i, x_{i+p}, a) < 1,$$

and hence we get the desired result, Thus, we have

$$S = \sum_{i=1}^{\infty} q^i \prod_{j=1}^{i} \theta(x_j, x_{n+p}, a)$$

with the partial sum

$$S_n = \sum_{i=1}^{\infty} q^i \prod_{j=1}^{i} \theta(x_j, x_{n+p}, a).$$

Consequently, we observe, for $n \leq 1,\ p \leq 1$, then

$$d_\theta(x_n, x_{n+p}, a) \leq q^n d_\theta(x_0, x_1, a)[S_{n+p-1} - S_{n-1}]. \qquad (3.4)$$

Letting $n \to \infty$ in (3.4), we conclude that the constructive sequence $\{x_n\}$ is Cauchy in the extended b_2-metric space (X, d_θ).

Theorem 3.3

Let $T,\ S$ be two self-mappings on a complete extended b_2-metric space (X, d_θ), such that that the pair $S,\ T$ form an $(\alpha,\ \beta)$-orbital-cyclic admissible pair. Suppose that

i. For each $x_0 \in X$

$$\lim_{n,m\to\infty} \theta(x_n, x_m, a) < \frac{1-k}{k},$$

where $x_{2n} = Sx_{2n-1}$ and $x_{2n+1} = Tx_{2n}$ for each $n \in \mathbb{N}$;

ii. there exists $\mathrm{x_0} \in \mathrm{X}$ such that $\alpha(\mathrm{x_0}, \mathrm{Tx_0}, \mathrm{a}) \geq 1, \forall\, a \in X$;

iii. either S and T are continuous, or

iv. if x_n is a sequence in X such that $x_n \to u$, then $\alpha(\mathrm{u}, \mathrm{Tu}, \mathrm{a}) \geq 1$ and $\beta(\mathrm{u}, \mathrm{Su}, \mathrm{a}) \geq 1$.

Moreover, if for all $x, y \in X$ and $k \in \left[0, \frac{1}{2}\right)$

$$\alpha(x, Tx, a)\beta(y, Sy, a)d_\theta(Tx, Sy, a) \leq k[d_\theta(x, Tx, a) + d_\theta(y, Sy, a)], \qquad (3.5)$$

then the pair of the mappings T, S possesses a common fixed point u, that is,

$Tu = u = Su.$

Proof

By assumption (ii), there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0, a) \geq 1$. Take $x_1 = Tx_0$ and $x_2 = Sx_1$. By induction, we construct a sequence $\{x_n\}$ such that

$$x_{2n} = Sx_{2n-1} \quad and \quad x_{2n+1} = Tx_{2n}, \forall n \in \mathbb{N}. \qquad (3.6)$$

We have $\alpha(x_0, x_1, a) \geq 1$, and since (S, T) is an α, β-orbital-cyclic admissible pair, we get

$$\alpha(x_0, x_1, a) \geq 1 \Longrightarrow \beta(Tx_0, STx_0, a) = \beta(x_1, x_2, a) \geq 1,$$

and

$$\beta(x\_1, x\_2, a) \geq 1 \Longrightarrow \alpha(Tx\_1, TSx\_1, a) = \alpha(x\_2, x\_3, a) \geq 1.$$

Applying again (iii),

$$\alpha(x_2, x_3, a) \geq 1 \Longrightarrow \beta(Tx_2, STx_2, a) = \beta(x_3, x_4, a) \geq 1$$

and

$$\beta(x_3, x_4, a) \geq 1 \Longrightarrow \alpha(Tx_3, TSx_3, a) = \alpha(x_4, x_5, a) \geq 1.$$

Recursively, we obtain

$$\alpha(x_{2n}, x_{2n+1}, a) \geq 1, \forall n \in \mathbb{N} \ , \qquad (3.7)$$

and

$$\beta(x_{2n+1}, x_{2n+2}, a) \geq 1, \ \forall n \in \mathbb{N}. \qquad (3.8)$$

Without loss of generality, we assume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}_0$.

Indeed, if $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}_0$, then $u = x_{n_0}$ forms a common fixed point for S and T, which finalizes the proof. More precisely, to see that u is the common fixed point for S and T, we shall examine the following two case. First, we assume that n_0 is even, that is, $n_0 = 2k$.

In this case, we have $x_{2k} = x_{2k+1} = Tx_{2k}$, that is, x_{2k} is a fixed point of T.

Now we shall prove that $x_{2k} = x_{2k+1} = Tx_{2k} = Sx_{2k}$.

Suppose on the contrary that

. By letting $x = x_{2k}$ and $y = x_{2k+1}$ in(3.5) and keeping in mind (3.7) and $d_\theta(Tx_{2k}, Sx_{2k+1}, a) > 0$ (3.8), we get that:

$$\begin{aligned} 0 < d_\theta(x_{2k+1}, x_{2k+2}, a) &= d_\theta(Tx_{2k}, Sx_{2k+1}, a) \\ &\leq \alpha(x_{2k}, Tx_{2k}, a)\beta(x_{2k+1}, Sx_{2k+1}, a)d_\theta(Tx_{2k}, Sx_{2k+1}, a) \\ &\leq k[d_\theta(x_{2k}, Tx_{2k}, a) + d_\theta(x_{2k+1}, Sx_{2k+1}, a)], \end{aligned}$$

which is a contradiction. Hence, we conclude that $d_\theta(Tx_{2k}, \ Sx_{2k+1}, \ a) = 0$,

and $x_{2k} = x_{2k+1} = Tx_{2k} = Sx_{2k+1}$, that is $x_{2k} = x_{2k+1} = u$ is a common fixed point of T and S. Second, we assume that n_0 is odd, that is, $n_0 = 2k - 1$.

$$x_n \neq x_{n+1} \text{for each } n \in \mathbb{N}_0 . \qquad (3.9)$$

In this case, we have $x_{2k-1} = x_{2k-1+1} = x_{2k} = Sx_{2k-1}$, that is, x_{2k-1} is fixed point of T.

Now we shall prove that

$$x_{2k-1} = x_{2k} = Sx_{2k-1} = Tx_{2k}.$$

Suppose on the contrary that $d_\theta(Tx_{2k}, Sx_{2k-1}, a) > 0$.

By letting $x = x_{2k-1}$ and $y = x_{2k}$ in (3.5) and keeping in mind (3.7) and (3.8) We get that

$$\begin{aligned} 0 < d_\theta(x_{2n+1}, x_{2n}, a) &= d_\theta(Tx_{2n}, Sx_{2n-1}, a) \\ &\leq \alpha(x_{2n}, Tx_{2n}, a)\beta(x_{2n-1}, Sx_{2n-1}, a)d_\theta(Tx_{2n}, Sx_{2n-1}, a) \\ &\leq k[d_\theta(x_{2n}, Tx_{2n}, a) + d_\theta(x_{2n-1}, Sx_{2n-1}, a)] \end{aligned}$$

$$= k[d_\theta(x_{2n}, x_{2n+1}, a) + d_\theta(x_{2n-1}, x_{2n}, a)], \tag{3.10}$$

and

$$d_\theta(x_{2n}, x_{2n+1}, a) \le q d_\theta(x_{2n-1}, x_{2n}, a) \tag{3.11}$$

for each $n \in \mathbb{N}_0$, where $q = \frac{k}{1-k} < 1$ with $k \in \left[0, \frac{1}{2}\right)$.

Combining (3.8) and (3.11), we can conclude that

$$d_\theta(x_m, x_{m+1}, a) \le q d_\theta(x_{m-1}, x_m, a) \tag{3.12}$$

for all $m \in \mathbb{N}$. From Lemma 3.2, taking into account (i),

$$\lim_{n,m\to\infty} \theta(x_n, x_m, a) < \frac{1-k}{k} = \frac{1}{q},$$

we obtain that $\{x_n\}$ is a Cauchy sequence.

By completeness of (X, d_θ), there is some point $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Naturally, we also have

$$x_{2n} \to u \quad and \quad x_{2n+1} \to u. \tag{3.13}$$

Due to the continuity of the mappings T and S, we get

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tu,$$

and

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Sx_n = S \lim_{n\to\infty} x_n = Su.$$

Let us consider now the alternative hypothesis (iv).

Taking $x = u$ and $y = x_{2n+1}$ in (3.5) and taking into account (3.8), we get

$$\begin{aligned} d_\theta(Tu, x_{2n+2}, a) &= d_\theta(Tu, Tx_{2n+1}, a) \\ &\le \alpha(u, Tu, a)\beta(d_\theta(Tu, Sx_{2n+1}, a)x_{2n+1}, Sx_{2n+1}, a) \\ &\le k[d_\theta(u, Tu, a) + d_\theta(x_{2n+1}, Sx_{2n+2}, a)]. \end{aligned} \tag{3.14}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} d_\theta(Tu, u, a) &= \lim_{n\to\infty} d_\theta(Tu, x_{2n+2}, a) \\ &\le k \lim_{n\to\infty} [d_\theta(u, Tu, a) + d_\theta(x_{2n+1}, x_{2n+2}, a)] \\ &= k d_\theta(u, Tu, a) < d_\theta(u, Tu, a), \end{aligned} \tag{3.15}$$

which implies $d_\theta(Tu, u, a) = 0$. Hence, we get that $Tu = u$.

Analogously, regarding (3.7) and (3.15), we observe that

$$d_\theta(x_{2n+1}, Su, a) = d_\theta(Tx_{2n}, Su, a)$$
$$\leq \alpha(x_{2n}, Tx_{2n}, a)\beta(u, Su, a)d_\theta(Tx_{2n}, Su, a)$$
$$\leq k[d_\theta(x_{2n}, Tx_{2n}, a) + d_\theta(u, Su, a)].$$

Now, letting $n \to \infty$ in the inequality above, we derive that

$$d_\theta(u, Su, a) = \lim_{n\to\infty} d_\theta(x_{2n+1}, Su, a)$$
$$\leq k \lim_{n\to\infty}[d_\theta(x_{2n}, Tx_{2n}, a) + d_\theta(u, Su, a)]$$
$$= kd_\theta(u, Su, a)$$
$$< d_\theta(u, Su, a),$$

Hence, we find that $Su = u$. Accordingly, we conclude that T and S have a common fixed point u.

Example 3.4

Let $X = [0,1]$ and $d_\theta: X \times X \times X \to [0, \infty)$ defined by

$$d_\theta(x, y, z) = \begin{cases} \frac{1}{xyz} & \text{for } x, y, z \in (0,1]\ x \neq y \neq z, \\ \frac{1}{xy} & \text{for } x, y \in (0,1]\ and\ z = 0, \\ 0 & \text{for } x, y, z \in (0,1] \text{ and at least two of } x, y \text{ and } z \text{ are equal} \end{cases}.$$

when

$$\theta(x, y, z) = \begin{cases} \frac{1 + x + y + z}{x + y + z} & \text{if } x, y, z \in (0,1], \\ 1 & \text{if } x = y = z = 0. \end{cases}$$

Then (X, d_θ) is an extended b_2-metric space.

Let $T: X \to X$ and $S: X \to X$ are defined as

$$T(x) = \begin{cases} 1, & \text{if } x = \frac{1}{2} \\ \frac{1}{2}, & \text{if } x = \frac{1}{4} \\ \frac{x+1}{2}, & \text{otherwise} \end{cases}, \qquad S(x) = \begin{cases} 1, & \text{if } x \in \left\{\frac{1}{4}, \frac{1}{2}\right\}, \\ x, & \text{otherwise} \end{cases}$$

Respectively, and two functions $\alpha, \beta: X \times X \times X \to [0, \infty)$ defined by

$$\alpha(x, y, a) = \begin{cases} 1, & \text{if } (x, y, a) \in \left\{\left(1,1,\frac{1}{3}\right), \left(\frac{1}{2}, 1, \frac{1}{3}\right), \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{3}\right)\right\}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\beta(x,y,a)=\begin{cases}1, & \text{if } (x,y,a)\in\left\{\left(1,1,\frac{1}{3}\right),\left(\frac{1}{2},1,\frac{1}{3}\right),\left(\frac{1}{4},1,\frac{1}{3}\right)\right\},\\ 0, & \text{otherwise}\end{cases}$$

we show that the pair $T,\ S$ forms an $(\alpha,\ \beta)$-orbital-cyclic admissible pair.

Indeed, for $x=1$,

$$\alpha(1,T1,a)=\alpha\left(1,1,\frac{1}{3}\right)\geq 1\rightarrow\beta(T1,ST1,a)=\beta\left(1,1,\frac{1}{3}\right)\geq 1,$$

and

$$\beta(1,S1,a)=\beta\left(1,1,\frac{1}{3}\right)\geq 1\rightarrow\alpha(S1,TS1,a)=\alpha\left(1,1,\frac{1}{3}\right)\geq 1.$$

For $x=\frac{1}{2}$,

$$\alpha\left(\frac{1}{2},T\frac{1}{2},a\right)=\alpha\left(\frac{1}{2},1,\frac{1}{3}\right)\geq 1\rightarrow\beta\left(T\frac{1}{2},ST\frac{1}{2},a\right)=\beta\left(1,1,\frac{1}{3}\right)\geq 1,$$

and

$$\beta\left(\frac{1}{2},S\frac{1}{2},a\right)=\beta\left(\frac{1}{2},1,\frac{1}{3}\right)\geq 1\rightarrow\alpha\left(S\frac{1}{2},TS\frac{1}{2},a\right)=\alpha\left(1,1,\frac{1}{3}\right)\geq 1.$$

For $x=\frac{1}{4}$,

$$\alpha\left(\frac{1}{4},T\frac{1}{4},a\right)=\alpha\left(\frac{1}{4},\frac{1}{2},\frac{1}{3}\right)\geq 1\rightarrow\beta\left(T\frac{1}{4},ST\frac{1}{4},a\right)=\beta\left(\frac{1}{2},1,\frac{1}{3}\right)\geq 1,$$

and

$$\beta\left(\frac{1}{4},S\frac{1}{4},a\right)=\beta\left(\frac{1}{4},1,\frac{1}{3}\right)\geq 1\rightarrow\alpha\left(S\frac{1}{4},TS\frac{1}{4},a\right)=\alpha\left(1,1,\frac{1}{3}\right)\geq 1.$$

We have thus proved that T is α orbital admissible and sure, because $\alpha\left(\frac{1}{4},T\frac{1}{4},a\right)\geq 1$ assumption(ii) is satisfied.

If $x_0\in\left\{\frac{1}{4},\frac{1}{2},1\right\}$, then $x_0=T^n x_0=1$, so

$$\lim_{n,m\rightarrow\infty}\theta(x_n,x_m,a)=\frac{3}{2}<3=\frac{1-k}{k},$$

where we choose $k=\frac{1}{3}<\frac{1}{2}$.

Otherwise. For each $x_0\in X-\left\{\frac{1}{4},\frac{1}{2},1\right\}$, we have

$$x_{2n-1} = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{n} + \frac{x_0}{2^n},\ \ X_{2n} = X_{2n-1}$$

and

$$\lim_{n\to\infty} X_n = 1.$$

So,

$$\lim_{n,m\to\infty} \theta(x_n, x_m, a) = \frac{3}{2} < 3 = \frac{1-k}{k}.$$

Hence, (i) is also verified.

We have

$$d_\theta\left(1, T1, \frac{1}{3}\right) = 0, \quad d_\theta\left(\frac{1}{2}, T\frac{1}{2}, \frac{1}{3}\right) = 6, \quad d_\theta\left(\frac{1}{4}, T\frac{1}{4}, \frac{1}{3}\right) = 24,$$

$$d_\theta\left(1, S1, \frac{1}{3}\right) = 0, \quad d_\theta\left(\frac{1}{2}, S\frac{1}{2}, \frac{1}{3}\right) = 6, \quad d_\theta\left(\frac{1}{4}, S\frac{1}{4}, \frac{1}{3}\right) = 12,$$

and

$$d_\theta\left(T1, S1, \frac{1}{3}\right) = 0, \quad d_\theta\left(T1, S\frac{1}{2}, \frac{1}{3}\right) = 0, \quad d_\theta\left(T1, S\frac{1}{4}, \frac{1}{3}\right) = 0,$$

$$d_\theta\left(T\frac{1}{2}, S1, \frac{1}{3}\right) = 0, \quad d_\theta\left(T\frac{1}{2}, S\frac{1}{2}, \frac{1}{3}\right) = 0, \quad d_\theta\left(T\frac{1}{2}, S\frac{1}{4}, \frac{1}{3}\right) = 0,$$

$$d_\theta\left(T\frac{1}{4}, S1, \frac{1}{3}\right) = 6, \quad d_\theta\left(T\frac{1}{4}, S\frac{1}{2}, \frac{1}{3}\right) = 6, \quad d_\theta\left(T\frac{1}{4}, S\frac{1}{4}, \frac{1}{3}\right) = 6.$$

Because in the other cases $\alpha(x, y, a) = 0$ and $\beta(x, y, a) = 0$.

It is enough to investigate the following situations:

Case (a): For $x \in \left\{1, \frac{1}{2}\right\}$ and $y \in \left\{1, \frac{1}{2}, \frac{1}{4}\right\}$ $a = \frac{1}{3}$. Then $\quad d_\theta(Tx, Sy, a) = 0.$

So inequality (3.5) is satisfied.

Case (b): Let $x = \frac{1}{4},\ y = 1,\ a = \frac{1}{3}$. Then

$$6 = d_\theta\left(T\frac{1}{4}, S1, a\right) = \alpha\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{3}\right)\beta\left(1,1,\frac{1}{3}\right) d_\theta\left(T\frac{1}{4}, S1, \frac{1}{3}\right)$$

$$\leq \frac{1}{3}\left[d_\theta\left(\frac{1}{4}, T\frac{1}{4}, \frac{1}{3}\right) + d_\theta\left(1, S1, \frac{1}{3}\right)\right] = \frac{1}{3}[24 + 0] = \frac{24}{3} = 8.$$

Case (c): Let $x = \frac{1}{4},\ y = \frac{1}{2},\ a = \frac{1}{3}$. Then

$$6 = d_\theta\left(T\frac{1}{4}, S\frac{1}{2}, a\right) = \alpha\left(\frac{1}{4}, T\frac{1}{4}, \frac{1}{3}\right)\beta\left(\frac{1}{2}, S\frac{1}{2}, \frac{1}{3}\right) d_\theta\left(T\frac{1}{4}, S\frac{1}{2}, \frac{1}{3}\right)$$

$$\leq \frac{1}{3}\left[d_\theta\left(\frac{1}{4}, T\frac{1}{4}, \frac{1}{3}\right) + d_\theta\left(\frac{1}{2}, S\frac{1}{2}, \frac{1}{3}\right)\right] = \frac{1}{3}[24+6] = \frac{30}{3} = 10.$$

Case (d): Let $x = \frac{1}{4},\ y = \frac{1}{4},\ a = \frac{1}{3}$. Then

$$6 = d_\theta\left(T\frac{1}{4}, S\frac{1}{4}, a\right) = \alpha\left(\frac{1}{4}, T\frac{1}{4}, \frac{1}{3}\right)\beta\left(\frac{1}{4}, S\frac{1}{4}, \frac{1}{3}\right)d_\theta\left(T\frac{1}{4}, S\frac{1}{4}, \frac{1}{3}\right)$$

$$\leq \frac{1}{3}\left[d_\theta\left(\frac{1}{4}, T\frac{1}{4}, \frac{1}{3}\right) + d_\theta\left(\frac{1}{4}, S\frac{1}{4}, \frac{1}{3}\right)\right] = \frac{1}{3}[24+12] = \frac{36}{3} = 12.$$

Therefore, all conditions of Theorem 3.3 are satisfied and t the pair of the mappings T, S possesses a common fixed point $u = 1$.

We will now examine the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-orbital-cyclic admissible mappings within the context of an extended $\boldsymbol{b_2}$-metric space.

Definition 3.5

Let $\boldsymbol{X}$ be a nonempty set $\boldsymbol{T} : \boldsymbol{X} \to \boldsymbol{X}$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} : \boldsymbol{X} \times \boldsymbol{X} \times \boldsymbol{X} \to [\boldsymbol{0}, \infty)$. We say that $\boldsymbol{T}$ is an $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-orbital-cyclic admissible mappings if

$$\left.\begin{array}{l} \boldsymbol{\alpha(x, Tx, a) \geq 1} \ \ \textbf{implies} \ \ \boldsymbol{\beta\left(Tx, T^2x, a\right) \geq 1}, \\ \boldsymbol{\beta(x, Tx, a) \geq 1} \ \ \textbf{implies} \ \ \boldsymbol{\alpha\left(Tx, T^2x, a\right) \geq 1}. \end{array}\right\} \qquad \textbf{(3.16)}$$

for all $\boldsymbol{x} \in \boldsymbol{X}$.

Corollary 3.6

Let $\boldsymbol{T}$ be a self-mapping on a complete extended $\boldsymbol{b_2}$-metric space $(\boldsymbol{X}, \boldsymbol{d_\theta})$, such that the mapping $\boldsymbol{T}$ form an $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-orbital-cyclic admissible mapping. Suppose that

i. i. For each $\boldsymbol{x_0} \in \boldsymbol{X}$,

$$\lim_{n,m\to\infty} \boldsymbol{\theta(x_n, x_m, a)} < \frac{\boldsymbol{1-k}}{\boldsymbol{k}},$$

where $\boldsymbol{x_n = T^n x_0, n = 1, 2}, \ldots$;

ii. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, a) \geq 1$ and $\beta(x_0, Tx_0, a) \geq 1$;

iii. either T is continuous, or

iv. if x_n is a sequence in X such that $x_n \to u$, then $\alpha(u, Tu, a) \geq 1$ and $\beta(u, Tu, a) \geq 1$.

Moreover, if for all $x, y \in X$ and $k \in \left[0, \frac{1}{2}\right)$

$$\alpha(x, Tx, a)\beta(y, Ty, a)d_\theta(Tx, Ty, a) \leq k[d_\theta(x, Tx, a) + d_\theta(y, Ty, a)], \tag{3.17}$$

then the pair of the mappings T possesses a fixed point u, that is, $Tu = u$.

on a theoretical basis.

Example 3.7

Let $X = \mathbb{R}$. Define $d_\theta: X \times X \times X \to [0, \infty)$ by

$$d_\theta(x, y, z) = \begin{cases} x^2 + y^2 + z^2, & x \neq y \neq z \neq x, \\ 0, & \text{if at least two of } x, y \text{ and } z \text{ are equal.} \end{cases}$$

And define $\theta: X \times X \times X \to [1, \infty)$

$$\theta(x, y, z) = |x| + |y| + |z| + 1.$$

Then (X, d_θ) is an extended b_2-metric space.

Let the self-map $T: X \to X$ be defined by

$$T(x) = \begin{cases} \dfrac{x}{8} & \text{if } x \in [0,1) \\ \sqrt{-x^2 + 3x - 2} & \text{if } x \in [1,2]. \end{cases}$$

Define also $\alpha,\ \beta \in X \times X \times X \to [0, \infty)$ by

$$\alpha(x, y, a) = \begin{cases} 2 & \text{if } x, y, a \in [0,1] \\ 0 & \text{otherwise,} \end{cases}, \qquad \beta(x, y, a) = \begin{cases} 1 & \text{if } x, y, a \in [0,1] \\ 2 & \text{if } x = 2, y = 0, a = \frac{1}{8} \\ 0 & \text{otherwise,} \end{cases}.$$

We show that T is (α, β)-orbital-cyclic-admissible.

Let $x, y, a \in X$ such that $\alpha(x, Tx, a) \geq 1$ and $\beta(x, Tx, a) \geq 1$. Then $x, y, a \in [0,1)$. On the other hand, if $x \in [0,1)$, then $Tx \leq 1$ and $T^2x \leq 1$.

It follows that $\alpha(Tx, T^2x, a) \geq 1$ and $\beta(Tx, T^2x, a) \geq 1$. Thus, the assumption holds.

For $x = 0$, we have $T0 = 0$ and $\alpha(0, T0, a) \geq 1$, respectively, $\beta(0, T0, a) \geq 1$, so assumption (ii) is satisfied. Let now $\{x_n\}$ be a sequence in X such that $x_n \to x$.

Then $\{x_n\} \subset [0,1]$ and $x, a \in [0,1]$. This implies that $\alpha(x, Tx, a) \geq 1$.

For $x_0 \in [0,1)$, we get $T^2x_0 = \frac{x_0}{8^n}$ and $\lim_{n,m\to\infty} \theta(T^n x_0, T^m x_0, a) = 1$.

If $x_0 \in [1,2]$, $Tx_0 \leq \frac{1}{4}$, and $\lim_{n,m\to\infty} \theta(T^n x_0, T^m x_0, a) = 1$. So assumption (i) is satisfied for $k = \frac{1}{3}$. We have the following cases:

Case (a). For $x, y \in [0,1)$ and $a = \frac{1}{8}$.

If $x \neq y \neq z$, then $d_\theta(x, y, z) = x^2 + y^2 + z^2$,

$$d_\theta(Tx, Ty, a) = \left(\frac{x}{8}\right)^2 + \left(\frac{y}{8}\right)^2 + \left(\frac{1}{8}\right)^2$$

$$= \frac{1}{64}(x^2 + y^2 + 1),$$

$$d_\theta(x, Tx, a) = x^2 + \left(\frac{x}{8}\right)^2 + \left(\frac{1}{8}\right)^2$$

$$d_\theta(y, Ty, a) = y^2 + \left(\frac{y}{8}\right)^2 + \left(\frac{1}{8}\right)^2$$

$$= \frac{1}{64}(65x^2 + 1), = \frac{1}{64}(65y^2 + 1),$$

Replaced in Corollary 3.6 we get

$$\alpha(x, Tx, a)\, \beta(y, Ty, a) d_\theta(Tx, Ty, a) \leq \frac{1}{3} \,.\, [d_\theta(x, Tx, a) + d_\theta(y, Ty, a)]$$

or

$$\frac{x^2 + y^2 + 1}{32} \leq \frac{1}{3}\left[\frac{1}{64}(65x^2 + 65y^2 + 2)\right] = \frac{65x^2 + 65y^2 + 2}{192},$$

which is true for any $x, y \in [0,1), a = \frac{1}{8}$.

If $x = y = z$, then $d_\theta(Tx, Ty, a) = 0$.

So, inequality (3.5) is satisfied, which is true for any $x, y \in [0,1)$.

Case (b). For $x = 1$, $y = 2$ and $a = \frac{1}{8}$.

We know that $\alpha(1, T1, a) = \alpha(1,0, a) \geq 1$ and $\beta(T1, T^2 1, a) = \beta(0,0, a) \geq 1$.

Also $\beta(2, T2, a) = \beta(2,0, a) \geq 1$ and $\alpha(T2, T^2 2, a) = \alpha(0,0, a) \geq 1$.

But in this case Corollary 3.6 is obvious, because $d_\theta(T1, T2, a) = 0$.

Case (c). For $x \in [0,1)$, $y = 2$ and $a = \frac{1}{8}$, Corollary 3.6 becomes

$$\alpha(x, Tx, a)\, \beta(2, T2, a) d_\theta(Tx, T2, a) \leq \frac{1}{3} \cdot [d_\theta(x, Tx, a) + d_\theta(2, T2, a)]$$

or

$$\frac{x^2 + 1}{16} \leq \frac{1}{3}\left[\frac{1}{64}(65x^2 + 1) + \frac{257}{64}\right] = \frac{65x^2 + 258}{192}.$$

Case (d). For all other cases, $\alpha(x, Tx, a) = 0$ or $\beta(x, Tx, a) = 0$, and for this reason inequality (3.17) holds. Therefore, all the conditions of Corollary 3.6 are satisfied and T has a fixed-point, $x = 0$.

Corollary 3.8

Let T be a self-mapping on a complete extended b_2-metric space (X, d_θ), such that T is an α-orbital admissible mapping. Suppose that

i. For each $x_0 \in X$,

$$\lim_{n,m\to\infty} \theta(x_n, x_m, a) < \frac{1-k}{k},$$

where $x_n = T^n x_0, n = 1,2, \ldots$;

ii. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, a) \geq 1$;

iii. either T is continuous, or

iv. if x_n is a sequence in X such that $x_n \to u$, then $\alpha(u, Tu, a) \geq 1$.

Moreover, if for all $x, y \in X$ and $k \in \left[0, \frac{1}{2}\right)$

$$\alpha(x, Tx, a)\alpha(y, Ty, a) d_\theta(Tx, Ty, a) \leq k[d_\theta(x, Tx, a) + d_\theta(y, Ty, a)], \tag{3.18}$$

then the pair of the mappings T possesses a fixed point u, that is, $Tu = u$.

Example 3.9

Let $X = \mathbb{R}$ be endower with extended b_2-metric space $d_\theta : X \times X \times X \to [0, \infty)$, defined by $d_\theta(x, y, z) = x^2 + y^2 + z^2$,

where, $\theta : X \times X \times X \to [1, \infty)$ defined by

$$\theta(x, y, z) = |x| + |y| + |z| + 1.$$

Let $T : X \longrightarrow X$ such that

$$T(x) = \begin{cases} \frac{x+1}{3} & \text{if } x \in [0,1] \\ \frac{x}{2} & \text{if } x \in (1,2] \end{cases}$$

Define also $\alpha, \in X \times X \times X \to [0, \infty)$ by

$$\alpha(x, y, a) = \begin{cases} 1 & \text{if } (x, y, a) \in \left\{\left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]\right\} \cup \left\{\left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right].\right\}, \\ 0 & \text{otherwise,} \end{cases}$$

we prove that Corollary 3.8 can be applied to T for $k = \frac{1}{4}$, but Theorem 3.3 cannot be applied to T. We show that T is an α-orbital admissible mapping. $x, y, a \in [0, \frac{1}{2}]$ then $Tx \leq \frac{1}{2}$ and $T^2 x \leq 1$.

Thus, $\alpha(x, Tx, a) \geq 1$ implies $\alpha(Tx, T^2 x, a) \geq 1$.

Similarly, we get that $\alpha(x, Tx, a) \geq 1$ implies $\alpha(Tx, T^2 x, a) \geq 1$ for all $x, y, a \in [\frac{1}{2}, 1]$, so T is an α-orbital admissible. In reason of the above arguments, $\alpha(0, T0, a) = \alpha\left(0, \frac{1}{3}, a\right) \geq 1$.

Thus, the assertion (ii) holds.

Note that, for each $x_0 \in X$, $T^n x_0 = \sum_{k=1}^{n} \left(\frac{1}{3}\right)^n + \frac{x_0}{3^n}$ and $\lim_{n \to \infty} T^n x_0 = \frac{1}{2}$. Hence, $\lim_{n,m \to \infty} \theta(T^n(x_0), T^m(x_0), a) = 2.\frac{1}{2} + 1 = 2 < 3 = \frac{1-k}{k}$.

So assumption (i) is satisfied, and because $\alpha\left(\frac{1}{2}, T\frac{1}{2}\right) = \alpha\left(\frac{1}{2}, \frac{1}{2}\right) \geq 1$, assumption (iv) is also satisfied. Let $x, y, \in \left[0, \frac{1}{2}\right]$ or $x, y \in \left[\frac{1}{2}, 1\right]$. We have

$$d_\theta(Tx, Ty, a) = \alpha(x, Tx, a)\, \alpha(y, Ty, a) d_\theta(Tx, Ty, a)$$

$$\left(\frac{x+1}{3}\right)^2 + \left(\frac{y+1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{x^2 + y^2 + 2x + 2y + 3}{9}.$$

And

$$k\,[d_\theta(x,Tx,a)+d_\theta(y,Ty,a)]=\frac{1}{4}\left[\frac{10x^2+2x+2}{9}+\frac{10y^2+2y+2}{9}\right]$$

$$=\frac{10x^2+\ 10y^2+2x+2y+4}{36}.$$

Replaced in inequality (3.3.3), we get

$$\frac{x^2+\ y^2+2x+2y+3}{9}\le\frac{10x^2+\ 10y^2+2x+2y+4}{36}$$

$$\frac{4x^2+\ 4y^2+8x+8y+12}{36}\le\frac{10x^2+\ 10y^2+2x+2y+4}{36}$$

or

$$4x^2+\ 4y^2+8x+8y+12\ \le 10x^2+\ 10y^2+2x+2y+4$$

$$6x^2+\ 6y^2-8x-8y-8\ge 0.$$

Hence, inequality (3.18) is satisfied. In other cases, inequality (3.18) is obviously satisfied, because $\alpha(\ x,y,a)=0$. Therefore, all conditions of Corollary 3.8 are satisfied and T has a unique fixed point, $x=\frac{1}{2}$.

4. Uniqueness of a fixed point

Notice that in this section, we are investigating the existence of common fixed points of certain operators. To ensure the uniqueness of a fixed point, we will consider the following hypothesis.

(A) For all $x,y\in \mathrm{CFix}(T)$, we have $\alpha(x,Tx,a)\ge 1$ and $\beta(y,Sy,a)\ge 1$.

Here, $\mathrm{CFix}(T)$ denotes the set of common fixed-point of T and S.

Theorem 4.1

Adding condition (A) to the hypotheses of Theorem 3.3, then u is the unique fixed-point of T.

Proof

Suppose on the contrary, that v is another fixed-point of T. From (A), there exists $v\in X$ such that

$$\alpha(x,Tx,a)\ge 1 \text{ and } \beta(y,Sy,a)\ge 1 \qquad (4.1).$$

Since T satisfies (3.5), we get that

$$d_\theta(Tu,Sv,a)\le\alpha(u,Tu,a)\beta(v,Sv,a)d_\theta(Tu,Sv,a)$$

$$\le k[d_\theta(u,Tu,a)+d_\theta(v,Sv,a)],$$

which yields that $d_\theta(u, v, a) \leq 0$.

Since the inequality above is possible only if $d_\theta(u, v, a) = 0$, that is, $u = v$. This is a contradiction. Thus, we proved that u is the unique fixed-point of T.

Conclusions

In this paper, we explored the presence of common fixed points for a specific mapping ((S, T) and (α, β) Orbital cyclic-admissible mapping), within the context of an extended b_2-metric space. The results we present encompass several well-known fixed-point theorems found in existing literature. Additionally, we provided some examples to demonstrate our results.

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