Odd Lindley Log Compound Rayleigh Distribution with Some Statistical Properties

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***Abstract*-In the current study a new family of distributions called the odd Lindley Log compound Rayleigh distribution is introduced and studied. Some statistical properties of the new distribution including moments, moment generating function and Renyi entropy are derived. Parameter estimations are obtained via the maximum likelihood method and the observed information matrix is derived. Finally, a real dataset is used to illustrate the importance and flexibility of the new proposed distribution.**

Keywords—Odd Lindley, Rayleigh, Distribution, Entropy

I. Introduction

Using probability distribution to represent real-life situations is considered one of the most important tasks of statistician. Modelling and interpreting lifetime data are seen as an essential in many practical situations, such as medical, business management, engineering, and finance. In recent decade, this has prompted academics to focus on developing families of probability distribution.

The compound Raleigh distribution plays a vital role for modelling and analysis in different areas of statistics, including reliability study, survival analysis and clinical medicine. Its statistical properties have been widely concerned by many statisticians and scholars. The compound Raleigh distribution is contained in three-parameter Burr Type XII distribution as a special case. For more details we refer the reader to Abushal[3], Shojaee et al.[15], Al-Hossain [5], Barot and Patal [7], Abd-Elmougod and Mahmoud [1], Reyad and Othman [13], Rasheed [11]. Rashed [12] introduced log compound Rayleigh distribution by logarithmic transformation to the random variable of compound Rayleigh distribution with its basic reliability properties, order statistics and maximum likelihood estimation. The cumulative distribution function (CDF), and the probability density function (PDF) of Log compound Raleigh distribution are given, respectively,

$$G(x;\theta,\lambda) = 1 - \lambda^{\theta}(\lambda + e^{2x})^{-\theta} \quad (1)$$

$$\mathrm{g}(x;\theta,\lambda) = 2\theta\lambda^{\theta}e^{2x}(\lambda + e^{2x})^{-(\theta+1)}, \qquad 0 < x < \infty \quad (2)$$

Gomes-Silva et al., [9] proposed odd Lindley-G family of distributions that generates distributions with greater flexibility in modelling of real-life data sets and reliability analysis. The CDF and PDF of the odd Lindley-G family distribution are respectively given as

$$F(x,\alpha,\xi) = \int_0^{\frac{G(x,\xi)}{1-G(x,\xi)}} \frac{\alpha^2}{(1+\alpha)}(1+x)e^{-\alpha x}dx = 1 - \frac{\alpha + (1 - G(x;\xi))}{(1+\alpha)(1 - G(x;\xi))} exp\left\{-\alpha\left(\frac{G(x,\xi)}{1-G(x,\xi)}\right)\right\} \quad (3)$$

$$f(x,\alpha,\xi) = \frac{\alpha^2}{(1+\alpha)}\frac{\mathrm{g}(x;\xi)}{(1-G(x;\xi))^3}\exp\left\{-\alpha\left(\frac{G(x,\xi)}{1-G(x,\xi)}\right)\right\} \quad (4)$$

Where $\mathrm{g}(x;\xi)$ is the PDF and $G(x;\xi)$ is the CDF of the baseline distribution to be extended depending on parameter vector ξ and $\alpha > 0$ is the shape parameter of the odd Lindley-G family.

Some distributions derived based on this approached, for example [Abouolmagd et al., [2], Ieren et al. [10], Atanda et al., [6], Samson et al. [14] and Bhat et al., [8]]

In the present study, we proposed an extension of log compound Rayleigh distribution called odd Lindley log compound Rayleigh (OLLCR) distribution based on odd Lindley-G family of generated distributions and provide some of its mathematical properties.

The rest of the article is organized as follows: In Section 2, we define the odd Lindley log compound Rayleigh distribution, we also moments and moments generating function are discussed in Section 3. In Section 4 we provide the Renyi and Shannon entropies. The maximum likelihood estimation of the parameter is obtained in Section 5. Real dataset is analyzed in Section 7. Finally, conclusion is presented in Section 7.

II. Odd Lindley Log Compound Rayleigh Distribution

In this we introduction the odd Lindley compound Rayleigh (OLLCR) distribution. Substituting equation (1) and (2) in equation (3) and (4) and simplifying, the CDF and PDF of the OLLCR distribution are obtained as:

$$F_{OLLCR}(x) = 1 - \frac{\alpha\lambda^{-\theta}(\lambda + e^{2x})^{\theta} + 1}{(1+\alpha)} \exp\left\{-\alpha\left(\frac{(\lambda + e^{2x})^{\theta}}{\lambda^{\theta}} - 1\right)\right\}; \quad \alpha,\theta,\lambda > 0 \tag{5}$$

and

$$f_{OLLCR}(x) = \frac{2\alpha^2\theta}{(1+\alpha)\lambda^{2\theta}} e^{2x}(\lambda + e^{2x})^{2\theta-1} \exp\left\{-\alpha\left(\frac{(\lambda + e^{2x})^{\theta}}{\lambda^{\theta}} - 1\right)\right\}; \quad \alpha,\theta,\lambda > 0 \tag{6}$$

Using power series expansion of exponential function, we obtain

$$\exp\left\{-\alpha\left(\frac{(\lambda + e^{2x})^{\theta}}{\lambda^{\theta}} - 1\right)\right\} = \sum_{i=0}^{\infty} \frac{(-1)^i\alpha^i}{i!}\left(\frac{(\lambda + e^{2x})^{\theta}}{\lambda^{\theta}} - 1\right)^i$$

and by using binomial expansion of $\left(\frac{(\lambda+e^{2x})^{\theta}}{\lambda^{\theta}} - 1\right)^i$ as given by

$$\left(\frac{(\lambda + e^{2x})^{\theta}}{\lambda^{\theta}} - 1\right)^i = \sum_{j=0}^{i} (-1)^j \binom{i}{j} \left(\frac{(\lambda + e^{2x})^{\theta}}{\lambda^{\theta}}\right)^{i-j}$$

Equation (6) becomes

$$f_{OLLCR}(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} \frac{2\theta\alpha^{i+2}(-1)^{i+j}}{(1+\alpha)\lambda^{\theta(i-j+2)}\, i!}\binom{i}{j}\, e^{2x}(\lambda + e^{2x})^{\theta(i-j+2)-1}$$

also, by using binomial expansion we get

$$\left(1 + \frac{e^{2x}}{\lambda}\right)^{\theta(i-j+2)-1} = \sum_{k=0}^{\infty} \binom{\theta(i-j+2)-1}{k} \left(\frac{e^{2x}}{\lambda}\right)^k$$

This OLLCR distribution pdf can be written in the form

$$f_{OLLCR}(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} \frac{2\theta\alpha^{i+2}(-1)^{i+j}}{(1+\alpha)i!}\left(\frac{1}{\lambda}\right)^{k+1}\binom{i}{j}\binom{\theta(i-j+2)-1}{k} e^{2x(k+1)} \tag{7}$$

Figure 1 and 2 are the plots of the PDF, CDF, reliability function and hazard rate function of the OLLCR distribution for different values of parameters.

The plots illustrate great flexibility of the OLLCR distribution.

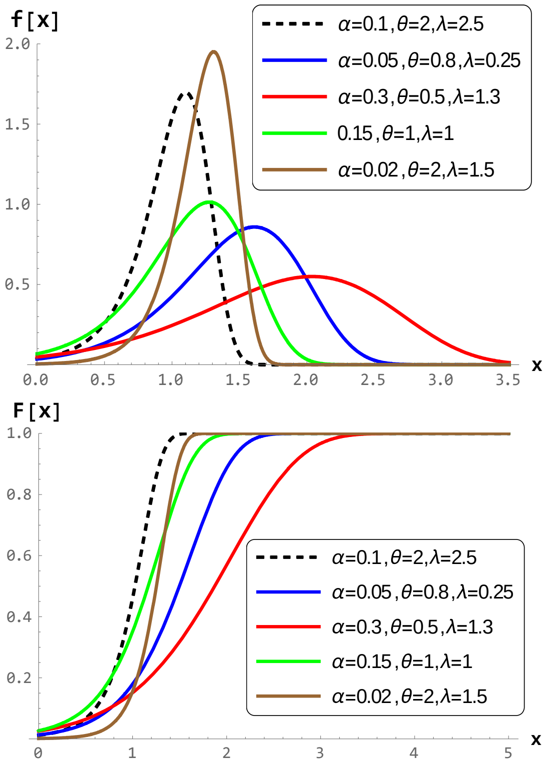

Fig. 1: The PDF and CDF of the OLLCR distribution for different values of parameters

The reliability (survival) function of the OLLCR distribution is given by:

$$R_{OLLCR}(x) = 1 - F_{OLLCR}(x) = \frac{\alpha\lambda^{-\theta}(\lambda + e^{2x})^{\theta} + 1}{(1+\alpha)} exp\left\{-\alpha\left(\frac{(\lambda + e^{2x})^{\theta}}{\lambda^{\theta}} - 1\right)\right\} \tag{8}$$

The hazard rate function is given by

$$h_{OLLCR}(x) = \frac{f_{OLLCR}(x)}{1 - F_{OLLCR}(x)} = \frac{2\alpha^2\theta e^{2x}(\lambda + e^{2x})^{2\theta-1}}{\lambda^{2\theta}(\alpha\lambda^{-\theta}(\lambda + e^{2x})^{\theta} + 1)} \tag{9}$$

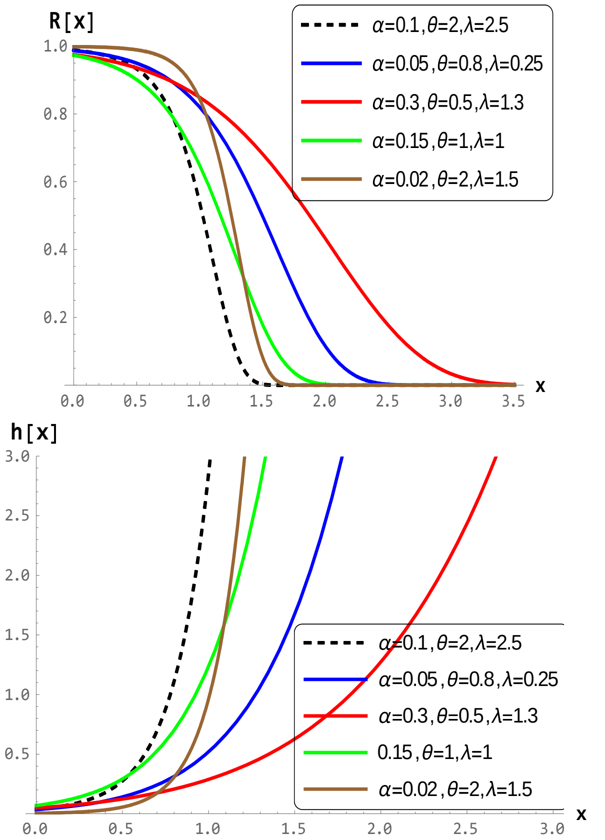

Fig. 2: The reliability and hazard function of the OLLCR distribution for different values of parameters

III. Moments and Moment Generating Function

A. *Moments*

Let X be a random variable that the OLLCR distribution, the r^{th} non-central moments is given by

$$\acute{\mu}_r = E(X^r) = \int_0^\infty x^r f(x)dx \qquad (10)$$

Using the pdf given in Equation (7)

$$\acute{\mu}_r = \frac{2\theta}{(1+\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\frac{\alpha^{i+2}(-1)^{i+j}}{i!}\left(\frac{1}{\lambda}\right)^{k+1}\binom{i}{j}\binom{\theta(i-j+2)-1}{k}\int_0^\infty x^r e^{2x(1+k)}dx \qquad (11)$$

Using the transformation $x = -\frac{y}{2(1+k)}$, we get

$$\acute{\mu}_r = \frac{2\theta}{(1+\alpha)} \times \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\frac{\alpha^{i+2}(-1)^{i+j}}{i!}\left(\frac{1}{\lambda}\right)^{k+1}\binom{i}{j}\binom{\theta(i-j+2)-1}{k}\left(\frac{-1}{2(k+1)}\right)^{r+1}\int_0^\infty y^r e^{-y}dy$$

where

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy.$$

$$\acute{\mu}_r = \frac{2\theta}{(1+\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\frac{\alpha^{i+2}(-1)^{i+j}}{i!}\left(\frac{1}{\lambda}\right)^{k+1}\binom{i}{j}\binom{\theta(i-j+2)-1}{k}\left(\frac{-1}{2(k+1)}\right)^{r+1}\Gamma(r+1) \qquad (12)$$

B. *Moments generating function*

The moments generating function of the random variable X with the pdf $f(x)$ is

$$\mu_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}\acute{\mu}_r \qquad (13)$$

Substituting from Equation (12) in to Equation (13), we obtain

$$\mu_x(t) = \frac{2\theta}{(1+\alpha)} \times \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\frac{t^r}{r!}\frac{\alpha^{i+2}(-1)^{i+j}}{i!}\left(\frac{1}{\lambda}\right)^{k+1}\binom{i}{j}\binom{\theta(i-j+2)-1}{k}\left(\frac{-1}{2(k+1)}\right)^{r+1}\Gamma(r+1) \qquad (14)$$

IV. Entropy

The Renyi entropy of random variable X with the $f(x)$ is defined is

$$I_R(\gamma) = \frac{1}{1-\gamma}log\left\{\int f^\gamma(x)dx\right\}, \qquad \gamma > 0, \quad \gamma \neq 1 \qquad (15)$$

Using the same approach for expanding the density

$$I_R(\gamma) = \frac{1}{1-\gamma}log\left\{\left(\frac{2\alpha^2\theta}{(1+\alpha)}\right)^\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\frac{(-1)^{i+j}(\gamma\alpha)^i}{i!}\left(\frac{1}{\lambda}\right)^{\gamma+k}\binom{i}{j}\binom{\theta(i-j+2\gamma)-\gamma}{k} \times \int_0^\infty e^{2(\gamma+k)x}dx\right\}$$

After simplifying the integral, then the Renyi entropy of OLLCR distribution is obtained as

$$I_R(\gamma) = \frac{1}{\gamma-1} \times log\left\{\left(\frac{2\alpha^2\theta}{(1+\alpha)}\right)^\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\frac{(-1)^{i+j}(\gamma\alpha)^i}{i!\,(2(\gamma+k))}\left(\frac{1}{\lambda}\right)^{\gamma+k}\binom{i}{j}\binom{\theta(i-j+2\gamma)-\gamma}{k}\right\} \qquad (16)$$

Moreover, the Shannon entropy defined by $E[-log f(x)]$ is the special case derived from $\lim_{\gamma\to1} I_R(\gamma)$. From equation (6),

$$E[-logf(x)] = -log\left(\frac{2\alpha^2\theta}{(1+\alpha)\lambda^{2\theta}}\right) - 2E(x) - (2\theta-1)E[log(\lambda+e^{2x})] + \alpha E\left(\frac{(\lambda+e^{2x})^\theta}{\lambda^\theta} - 1\right)$$

Using the expansion $\log(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$

$$E[log(\lambda+e^{2x})] = log(\lambda) + E\left[log\left(1+\frac{e^{2x}}{\lambda}\right)\right]$$

$$E\left[log\left(1+\frac{e^{2x}}{\lambda}\right)\right]=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\lambda^n}E(e^{2nx})$$

$$E\left(\frac{(\lambda+e^{2x})^{\theta}}{\lambda^{\theta}}-1\right)=E\left[\left(1+\frac{e^{2x}}{\lambda}\right)^{\theta}-1\right]$$

Similarly, using the expansion $(1+x)^v=\sum_{n=0}^{\infty}\binom{v}{n}x^n$

$$E\left[\left(1+\frac{e^{2x}}{\lambda}\right)^{\theta}\right]=\sum_{n=0}^{\infty}\frac{1}{\lambda^n}\binom{\theta}{n}E(e^{2nx})$$

The Shannon entropy can be written as

$$E[-logf(x)]=-log\left(\frac{2\alpha^2\theta}{(1+\alpha)\lambda^{2\theta}}\right)-2E(x)-(2\theta-1)\left(log(\lambda)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\lambda^n}E(e^{2nx})\right)+\alpha\left(\sum_{n=0}^{\infty}\frac{1}{\lambda^n}\binom{\theta}{n}E(e^{2nx})-1\right)$$

Now,

$$E(x)=\frac{2\theta}{(1+\alpha)}\times\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\frac{\alpha^{i+2}(-1)^{i+j}}{i!}\left(\frac{1}{\lambda}\right)^{k+1}\binom{i}{j}\binom{\theta(i-j+2)-1}{k}\left(\frac{-1}{2(k+1)}\right)^2$$

$$E(e^{2nx})=-\frac{2\theta}{(1+\alpha)}\times\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\frac{\alpha^{i+2}(-1)^{i+j}}{i!}\left(\frac{1}{\lambda}\right)^{k+1}\binom{i}{j}\binom{\theta(i-j+2)-1}{k}\left(\frac{1}{2(n+k+1)}\right)$$

V. Maximum Likelihood Estimation

Let $X_1, X_2, \dots, X_n$ be a random sample from OLLCR distribution. Then its likelihood function is

$$L=\prod_{i=1}^{n}f(x,\alpha,\theta,\lambda)$$

$$=\left(\frac{2\alpha^2\theta}{(1+\alpha)\lambda^{2\theta}}\right)^n e^{2\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}(\lambda+e^{2x})^{2\theta-1}exp\left\{-\alpha\sum_{i=1}^{n}\left(\frac{(\lambda+e^{2x})^{\theta}}{\lambda^{\theta}}-1\right)\right\}$$

The log likelihood function can be expressed as

$$\ell=logL=nlog(2)+2n\,log(\alpha)+nlog(\theta)-nlog(1+\alpha)-2n\theta log(\lambda)+2\sum_{i=1}^{n}x_i+(2\theta-1)\sum_{i=1}^{n}log(\lambda+e^{2x_i})-\alpha\sum_{i=1}^{n}\left(\frac{(\lambda+e^{2x_i})^{\theta}}{\lambda^{\theta}}-1\right)\quad(17)$$

differentiating partially with respect parameters α, θ and λ in Equation (17),

$$\frac{\partial\ell}{\partial\alpha}=\frac{2n}{\alpha}-\frac{n}{(1+\alpha)}-\sum_{i=1}^{n}\left(\frac{(\lambda+e^{2x_i})^{\theta}}{\lambda^{\theta}}-1\right)$$

$$\frac{\partial\ell}{\partial\theta}=\frac{n}{\theta}-2nlog(\lambda)+2\sum_{i=1}^{n}log(\lambda+e^{2x_i})-\alpha\sum_{i=1}^{n}\lambda^{-\theta}(\lambda+e^{2x_i})^{\theta}[log(\lambda+e^{2x_i})-log(\lambda)]$$

$$\frac{\partial\ell}{\partial\lambda}=-\frac{2n\theta}{\lambda}+(2\theta-1)\sum_{i=1}^{n}\frac{1}{\lambda+e^{2x_i}}-\alpha\sum_{i=1}^{n}\theta\lambda^{-\theta}(\lambda+e^{2x_i})^{\theta}[(\lambda+e^{2x_i})^{-1}-\lambda^{-1}]$$

The maximum likelihood estimators of the parameters α, θ and λ using nonlinear equations above, we equate equations to zero and solve them simultaneously.

In order to construct confidence intervals for the parameter and hypothesis testing on the model parameters, we require 3×3 observed information matrix $\mathcal{J}(\zeta)$ is given by

$$\mathcal{J}(\zeta)=\begin{pmatrix}\frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\theta} & \frac{\partial^2\ell}{\partial\alpha\partial\lambda}\\ \frac{\partial^2\ell}{\partial\theta\partial\alpha} & \frac{\partial^2\ell}{\partial\theta^2} & \frac{\partial^2\ell}{\partial\theta\partial\lambda}\\ \frac{\partial^2\ell}{\partial\lambda\partial\alpha} & \frac{\partial^2\ell}{\partial\lambda\partial\theta} & \frac{\partial^2\ell}{\partial\lambda^2}\end{pmatrix},$$

where

$$\frac{\partial^2\ell}{\partial\alpha^2}=-\frac{2n}{\alpha^2}+\frac{n}{(1+\alpha)^2}$$

$$\frac{\partial^2\ell}{\partial\theta^2}=-\frac{n}{\theta^2}-\alpha\sum_{i=1}^{n}\lambda^{-\theta}(\lambda+e^{2x_i})^{\theta}\left(\log(\lambda+e^{2x_i})-\log(\lambda)\right)^2$$

$$\frac{\partial^2\ell}{\partial\lambda^2}=\frac{2n\theta}{\lambda^2}-(2\theta-1)\sum_{i=1}^{n}\frac{1}{(\lambda+e^{2x_i})^2}-\alpha\sum_{i=1}^{n}\theta\lambda^{-\theta}(\lambda+e^{2x_i})^{\theta}\left((\theta+1)\lambda^{-2}+(\theta-1)(\lambda+e^{2x_i})^{-2}-2\theta\lambda^{-1}(\lambda+e^{2x_i})^{-1}\right)$$

$$\frac{\partial^2\ell}{\partial\alpha\partial\theta}=-\sum_{i=1}^{n}\lambda^{-\theta}(\lambda+e^{2x_i})^{\theta}\left(\log(\lambda+e^{2x_i})-\log(\lambda)\right)$$

$$\frac{\partial^2\ell}{\partial\alpha\partial\lambda}=-\sum_{i=1}^{n}\theta\lambda^{-\theta}(\lambda+e^{2x_i})^{\theta}((\lambda+e^{2x_i})^{-1}-\lambda^{-1})$$

$$\frac{\partial^2\ell}{\partial\theta\partial\lambda}=-\frac{2n}{\lambda}+2\sum_{i=1}^{n}\frac{1}{\lambda+e^{2x_i}}-\alpha\sum_{i=1}^{n}\lambda^{-\theta}(\lambda+e^{2x_i})^{\theta}\left(((\lambda+e^{2x_i})^{-1}-\lambda^{-1})\{1+\theta\left(\log(\lambda+e^{2x_i})-\log(\lambda)\right)\}\right)$$

VI. Application

In the section, we provide an application of the OLLCR distribution to real dataset. The dataset represents the time to failure (103h) of turbocharger of on type of engine used by Xu et al. [18], Afify et al. [4] and Tushar et al. [17]. The data is: 1.6, 3.5, 4.8, 5.4, 6.0, 6.5, 7.0, 7.3, 7.7, 8.0, 8.4, 2.0, 3.9, 5.0, 5.6, 6.1, 6.5, 7.1, 7.3, 7.8, 8.1, 8.4, 2.6, 4.5, 5.1, 5.8, 6.3, 6.7, 7.3, 7.7, 7.9, 8.3, 8.5, 3.0, 4.6, 5.3, 6.0, 8.7, 8.8, 9.0.

In the following, our proposed model is compared with of Log compound Raleigh (LCR) and Lindley (L) distributions. The parameter estimates are obtained using the maximum likelihood method, then we present comparison criteria values: Akaike information criterion (AIC), corrected Akaike information criterion (AICC) and Bayesian information criterion (BIC) defined as:

$$\mathrm{AIC} = 2k - 2\ell,$$

$$\mathrm{AICC} = \mathrm{AIC} + \frac{2k(k+1)}{n-k-1},$$

$$\mathrm{BIC} = \ k \log n \ - 2\ell$$

where k is the number of parameters, n the sample size and ℓ is the maximized of the log-likelihood function under the considered model. From Table 1, it is obvious that the OLLCR distribution has the lowest values for the AIC, AICC and BIC statistics among the fitted models. So, the OLLCR distribution could be chosen as the best model. Figure 3 depict the data histogram with calculated PDF curves and empirical CDF curves. The OLLCR distri-bution provides better adequate to these data

Table 1: MLEs of the parameters, AIC, AICC and BIC values for data set

Model	MLEs	-ℓ	AIC	AICC	BIC
L	$\hat{\theta} = 0.284453$	104.285	210.57	210.675	212.259
LCR	$\hat{\theta} = 0.228681$ $\hat{\lambda} = 5871.95$	93.1255	190.251	190.575	193.629
OLLCR	$\hat{\alpha} = 0.162533$ $\hat{\theta} = 0.206674$ $\hat{\lambda} = 4.82551$	80.5692	167.138	167.805	172.205

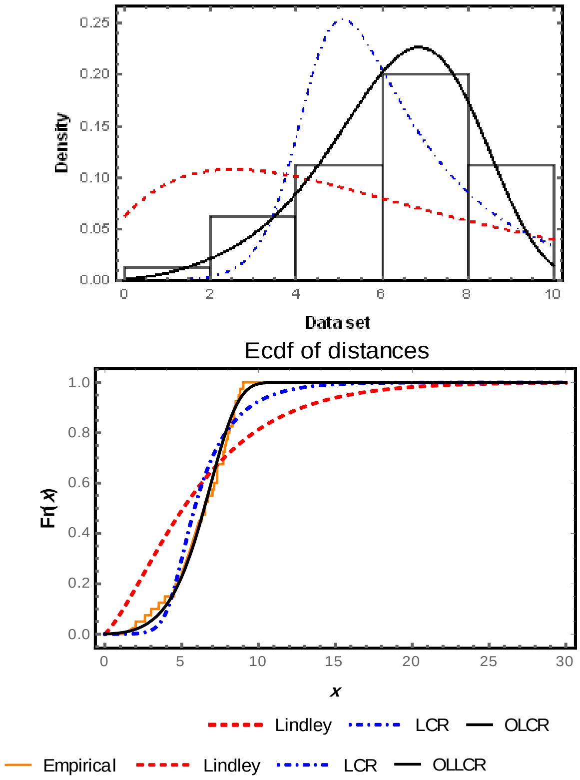

Fig 3: Estimated densities and cumulative densities of the models for dataset

VII. CONCLUSION

In this study, we introduce generalization of the log compound Rayleigh distribution called odd Lindley log compound Rayleigh distribution and presented its theoretical properties. The estimation of the distribution Parameters is performed using Maximum likelihood method. This distribution has been applied to real dataset; the results show that the odd Lindley log compound Rayleigh distribution provides a significantly better fit than other models.

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