Bayesian Estimation for the Parameters of the Cosine Inverse Log Compound Rayleigh Distribution

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Abstract- **In this paper, we consider the Bayesian estimation of the parameters and reliability function for a Cosine inverse log compound Rayleigh distribution under squared error and squared logarithmic loss functions. We use Lindley's approximation to compute the Bayesian estimates. This method is evaluated using mean square error through simulation study with varying sample size.**

Keywords— Cosine-G, Inverse Log Compound Rayleigh Distribution, Bayesian estimation, Lindley Approximation

I. INTRODUCTION

Statistical distributions are fundamental tools for describing real-world phenomena. Numerous classical distributions have been extensively used over the past decades for modelling data in several areas. There is always a clear need for extending and modifying the existed forms of such distributions. Such extension or modification are commonly achieved by combing classical distributions with generalized ones. Sometimes, it is also preferable to add one parameter or more to the well-known classical parametric distributions which is considered more flexible in modelling data. The primary aim of this research is focusing on the family of cosine distributions developed by Souza et al. [10]. Several notable trigonometric distributions have been introduced in the literature, for instance, one can consult Tomy et al. [11], Isa et al., [3, 4, 5, 6] and Mustapha et al., [8].

The CDF and PDF of the Cosine-G family are respectively given as:

$$F(x;\xi) = 1 - Cos\left\{\frac{\pi}{2}G(x;\xi)\right\} \quad (1)$$

$$f(x;\xi) = \frac{\pi}{2}g(x;\xi)Sin\left\{\frac{\pi}{2}G(x;\xi)\right\} \quad (2)$$

where $g(x;\xi)$ and $G(x;\xi)$ are the PDF and CDF of the baseline distribution.

Rasheed [9] introduced log compound Rayleigh distribution by logarithmic transformation to the random variable of compound Rayleigh distribution with its basic reliability properties, order statistics and maximum likelihood estimation. The proposed model will be named inverse log Compound Rayleigh (ILCR) distribution. Its CDF and PDF are given respectively by:

$$G(x;\theta,\lambda) = \lambda^{\theta}(\lambda + e^{2/x})^{-\theta} \quad (3)$$

$$g(x;\xi) = 2\theta\lambda^{\theta}x^{-2}e^{2/x}(\lambda + e^{2/x})^{-(\theta+1)} \quad (4)$$

Then the reliability of ILCR distribution is given by:

$$R(t) = 1 - \lambda^{\theta}\left(\lambda + e^{\frac{2}{t}}\right)^{-\theta};\, t > 0 \quad (5)$$

Recently, Bayesian estimation approach has receivable great attention by a numerous of researchers. Bayes analysis is an important approach in statistical modelling, which formally seeking to use of prior information and Bayes theorem provides the formal basis for utilizing this information. This paper proposes the Bayesian estimation procedures for the unknown parameters of Cosine inverse log compound Rayleigh distribution by using Lindley's approximation technique.

The remainder of this paper is structured as follows: in Section 2, we introduced Cosine inverse log compound Rayleigh distribution. The Bayes estimates of the unknown parameters are obtained in Section 3 using Lindley's approximation. Simulation studies are presented in Section 4. Finally, we conclude the paper in Section 5.

II. COSINE INVERSE LOG COMPOUND RAYLEIGH DISTRIBUTION

In this section, the Cosine inverse log compound Rayleigh distribution (CILCR) has been introduced. The CDF of ILCR distribution can be obtained by inserting Equation (3) in Equation (1) and is given by:

$$F(x) = 1 - Cos\left\{\frac{\pi}{2}\lambda^{\theta}\left(\lambda + e^{\frac{2}{x}}\right)^{-\theta}\right\};\ \theta,\lambda > 0 \quad (6)$$

and the corresponding PDF and reliability function, respectively are given by

$$f(x/\theta,\lambda) = \pi\theta\lambda^{\theta}x^{-2}e^{2/x}\left(\lambda + e^{\frac{2}{x}}\right)^{-(\theta+1)} sin\left\{\frac{\pi}{2}\lambda^{\theta}\left(\lambda + e^{\frac{2}{x}}\right)^{-\theta}\right\};\ \theta,\lambda > 0 \quad (7)$$

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and

$$R(t) = Cos\left\{\frac{\pi}{2}\lambda^{\theta}\left(\lambda + e^{\frac{2}{t}}\right)^{-\theta}\right\}; \ \theta, \lambda > 0 \tag{8}$$

Let $\boldsymbol{x} = (X_1, X_2, \dots, X_n)$ be a random vector having probability density function (7). Then the likelihood function is given by:

$$L(\theta, \lambda/\boldsymbol{x}) = \left(\pi\theta\lambda^{\theta}\right)^{n} \ e^{\sum_{i=1}^{n} 2/x_i} \prod_{i=1}^{n} \left\{ x_i^{-2}\left(\lambda + e^{\frac{2}{x_i}}\right)^{-(\theta+1)} Sin\left\{\frac{\pi}{2}\eta_i\right\} \right\}, \tag{9}$$

where

$$\eta_i(\theta, \lambda) = \lambda^{\theta}\left(\lambda + e^{\frac{2}{x_i}}\right)^{-\theta} \ ; i = 1,2, \dots, n.$$

For simplification we may use η_i instead of $\eta_i(\theta, \lambda)$.

III. Bayesian Estimation

In this section, we have obtained the Bayes estimates for the unknown parameters θ and λ, and reliability function. Squared error and squared logarithmic loss functions are used. We assume that θ and λ are independently distribution as Gamma (a, b) and Gamma (c, d) priors. Therefore, the joint prior is:

$$g(\theta, \lambda) \propto g_1(\theta) g_2(\lambda), \tag{10}$$

where

$$g_1(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}; \quad a, b > 0 \tag{11}$$

$$g_2(\lambda) = \frac{d^c}{\Gamma(c)}\lambda^{c-1}e^{-d\lambda}; \quad c, d > 0 \tag{12}$$

Based on the likelihood function (9) and the joint prior density (10), the joint posterior is given as

$$P(\theta, \lambda/\boldsymbol{x}) = \frac{L(\theta,\lambda/x)g(\theta,\lambda)}{\int_0^{\infty}\int_0^{\infty} L(\theta,\lambda/x)g(\theta,\lambda)\, d\theta\, d\lambda} \tag{13}$$

Therefore, the Bayes estimate of any function of θ and λ, say $u(\theta, \lambda)$ under the squared error loss function (SELF) is

$$\hat{u}_{BS}(\theta, \lambda) = E_{\theta,\lambda/x}[u(\theta, \lambda)] = \frac{\int_0^{\infty}\int_0^{\infty} u(\theta,\lambda)L(\theta,\lambda/x)g(\theta,\lambda)\, d\theta\, d\lambda}{\int_0^{\infty}\int_0^{\infty} L(\theta,\lambda/x)g(\theta,\lambda)\, d\theta\, d\lambda} \tag{14}$$

Also, the Bayes estimate of $u(\theta, \lambda)$ using squared logarithmic loss function (SLLF) which is proposed by Brown [1] and has been studied by Feroze and Aslam [2] is

$$\hat{u}_{BS}(\theta, \lambda) = exp\left[E_{\theta,\lambda/x}[\log(u(\theta, \lambda))]\right], \tag{15}$$

where

$$E_{\theta,\lambda/x}[\log(u(\theta, \lambda))] = \frac{\int_0^{\infty}\int_0^{\infty} \log(u(\theta,\lambda))L(\theta,\lambda/\boldsymbol{x})g(\theta,\lambda)\, d\theta\, d\lambda}{\int_0^{\infty}\int_0^{\infty} L(\theta,\lambda/\boldsymbol{x})g(\theta,\lambda)\, d\theta\, d\lambda} \tag{16}$$

It may be noted here that (14) and (16) do not simplify to nice closed form. In this case Lindley's approximation can be used to obtain the Bayes estimators for the parameters.

3.1 Lindley Approximation Method

Lindley [7] proposed a procedure to approximate the integrals usually occurred in Bayes estimator, which includes the posterior expectation is expressible in the form of ratio of integral as follow

$$I(\boldsymbol{x}) = E(u(\theta, \lambda)/\boldsymbol{x}) = \frac{\int_{(\theta,\lambda)} u(\theta,\lambda)e^{L(\theta,\lambda)+G(\theta,\lambda)}d(\theta,\lambda)}{\int_{(\theta,\lambda)} e^{L(\theta,\lambda)+G(\theta,\lambda)}d(\theta,\lambda)}, \tag{17}$$

where $u(\theta, \lambda)$ is a function of θ and λ only, $L(\theta, \lambda)$ is the log-likelihood function and $G(\theta, \lambda)$ is log of joint prior density. The ratio of integral of the from (17) can be approximation as

$$I(x) = u(\theta, \lambda) + \frac{1}{2}[(u_{11} + 2u_1\rho_1)\sigma_{11} + (u_{21} + 2u_2\rho_1)\sigma_{21} + (u_{12} + 2u_1\rho_2)\sigma_{12} + (u_{22} + 2u_2\rho_2)\sigma_{22}] + \frac{1}{2}[(u_1\sigma_{11} + u_2\sigma_{12})(L_{111}\sigma_{11} + L_{121}\sigma_{12} + L_{211}\sigma_{21} + L_{221}\sigma_{22}) + (u_1\sigma_{21} + u_2\sigma_{22})(L_{211}\sigma_{11} + L_{122}\sigma_{12} + L_{212}\sigma_{21} + L_{222}\sigma_{22})] \tag{18}$$

Here

$u_1 = \frac{\partial u(\theta,\lambda)}{\partial\theta}$, $u_2 = \frac{\partial u(\theta,\lambda)}{\partial\lambda}$, $u_{11} = \frac{\partial^2 u(\theta,\lambda)}{\partial\theta^2}$, $u_{12} = \frac{\partial^2 u(\theta,\lambda)}{\partial\theta\partial\lambda}$, $u_{22} = \frac{\partial^2 u(\theta,\lambda)}{\partial\lambda^2}$, $\rho_1 = \frac{\partial G(\theta,\lambda)}{\partial\theta}$, $\rho_2 = \frac{\partial G(\theta,\lambda)}{\partial\lambda}$, $L_{11} = \frac{\partial^2 L(\theta,\lambda)}{\partial\theta^2}$, $L_{12} = \frac{\partial^2 L(\theta,\lambda)}{\partial\theta\partial\lambda}$, and $L_{22} = \frac{\partial^2 L(\theta,\lambda)}{\partial\lambda^2}$, $L_{111} = \frac{\partial^3 L(\theta,\lambda)}{\partial\theta^3}$, $L_{112} = \frac{\partial^3 L(\theta,\lambda)}{\partial\theta^2\partial\lambda}$, $L_{122} = \frac{\partial^3 L(\theta,\lambda)}{\partial\theta\partial\lambda^2}$, $L_{222} = \frac{\partial^3 L(\theta,\lambda)}{\partial\lambda^3}$ and $\sigma_{ij} = \left(-\frac{1}{L_{ij}}\right), (i,j), i = 1,2.$

we have

$$\rho_1 = \frac{(a-1)}{\theta} - b \quad \text{and} \quad \rho_2 = \frac{(c-1)}{\lambda} - d$$

$$L(\theta, \lambda) = n\log\pi + n\log\theta + n\theta\log\lambda + \sum_{i=1}^{n}\frac{2}{x_i} - 2\sum_{i=1}^{n}\log x_i - (\theta+1)\sum_{i=1}^{n}\log\left(\lambda + e^{\frac{2}{x_i}}\right) + \sum_{i=1}^{n}\log Sin\left\{\frac{\pi}{2}\eta_i\right\} \tag{19}$$

From (19), we obtain the quantities

$$L_1 = \frac{n}{\theta} + n\log\lambda - \sum_{i=1}^{n}\log\left(\lambda + e^{\frac{2}{x_i}}\right) + \sum_{i=1}^{n}\cot\left\{\frac{\pi}{2}\eta_i\right\}\left\{\frac{\pi}{2}\eta_i\left(\log\lambda - \log\left(\lambda + e^{\frac{2}{x_i}}\right)\right)\right\}$$

$$L_2 = \frac{n\theta}{\lambda} - (\theta+1)\sum_{i=1}^{n}\frac{1}{\left(\lambda + e^{\frac{2}{x_i}}\right)} + \sum_{i=1}^{n}\cot\left\{\frac{\pi}{2}\eta_i\right\}\left\{\frac{\pi}{2}\theta\eta_i\left(\lambda^{-1} - \left(\lambda + e^{\frac{2}{x_i}}\right)^{-1}\right)\right\}$$

$$L_{11} = -\frac{n}{\theta^2} + \sum_{i=1}^{n}\left(\cot\left\{\frac{\pi}{2}\eta_i\right\} - \left(\frac{\pi}{2}\eta_i\csc^2\left\{\frac{\pi}{2}\eta_i\right\}\right)\right)\left(\frac{\pi}{2}\eta_i\left(\log\lambda - \log\left(\lambda + e^{\frac{2}{x_i}}\right)\right)^2\right)$$

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$$
\begin{aligned}
L_{12} = L_{21} = \frac{n}{\lambda} - \sum_{i=1}^{n} \frac{1}{\left(\lambda + e^{\frac{2}{x_i}}\right)}
&+ \sum_{i=1}^{n} \Bigg(\cot\left\{\frac{\pi}{2}\eta_i\right\} \Bigg(\frac{\pi}{2}\eta_i \Bigg(\lambda^{-1} \\
&- \left(\lambda + e^{\frac{2}{x_i}}\right)^{-1} \Bigg) \Bigg(1 \\
&+ \theta \log\lambda - \theta \log\left(\lambda + e^{\frac{2}{x_i}}\right) \Bigg) \Bigg) \Bigg) \\
&- \Bigg(\csc^2\left\{\frac{\pi}{2}\eta_i\right\} \theta \left\{\frac{\pi}{2}\eta_i\right\}^2 \Bigg(\lambda^{-1} \\
&- \left(\lambda + e^{\frac{2}{x_i}}\right)^{-1} \Bigg) \left(\log\lambda - \log\left(\lambda + e^{\frac{2}{x_i}}\right) \right) \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
L_{22} = -\frac{n\theta}{\lambda^2} + (\theta + 1) \sum_{i=1}^{n} \frac{1}{\left(\lambda + e^{\frac{2}{x_i}}\right)^2}
&+ \sum_{i=1}^{n} \frac{\pi}{2}\theta\eta_i \Bigg(\cot\left\{\frac{\pi}{2}\eta_i\right\} \Bigg(\theta \Bigg(\lambda^{-1} \\
&- \left(\lambda + e^{\frac{2}{x_i}}\right)^{-1} \Bigg)^2 - \lambda^{-2} + \left(\lambda + e^{\frac{2}{x_i}}\right)^{-2} \Bigg) \\
&- \csc^2\left\{\frac{\pi}{2}\eta_i\right\} \Bigg(\frac{\pi}{2}\theta\eta_i \Bigg(\lambda^{-1} \\
&- \left(\lambda + e^{\frac{2}{x_i}}\right)^{-1} \Bigg)^2 \Bigg) \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
L_{111} = \frac{2n}{\theta^3} + \sum_{i=1}^{n} \Bigg(\log\lambda
&- \log\Bigg(\lambda \\
&+ e^{\frac{2}{x_i}} \Bigg) \Bigg)^3 \left(2\left(\frac{\pi}{2}\eta_i\right)^3 \cot\left\{\frac{\pi}{2}\eta_i\right\} \csc^2\left\{\frac{\pi}{2}\eta_i\right\} \right. \\
&\left. - \frac{3}{4}\csc^2\left\{\frac{\pi}{2}\eta_i\right\} (\pi\eta_i)^2 + \cot\left\{\frac{\pi}{2}\eta_i\right\} \left(\frac{\pi}{2}\eta_i\right) \right)
\end{aligned}
$$

$$
\begin{aligned}
L_{112} = L_{211} = \sum_{i=1}^{n} \pi\theta\eta_i \cot\left\{\frac{\pi}{2}\eta_i\right\} \csc^2\left\{\frac{\pi}{2}\eta_i\right\} \Bigg(\lambda^{-1}
&- \Bigg(\lambda \\
&+ e^{\frac{2}{x_i}} \Bigg)^{-1} \Bigg) \left(\frac{\pi}{2}\eta_i \left(\log\lambda - \log\left(\lambda + e^{\frac{2}{x_i}}\right) \right) \right)^2 \\
&+ \pi\eta_i \left(\log\lambda - \log\left(\lambda + e^{\frac{2}{x_i}}\right) \right) \Bigg(\lambda^{-1} \\
&- \left(\lambda + e^{\frac{2}{x_i}}\right)^{-1} \Bigg) \Bigg(\cot\left\{\frac{\pi}{2}\eta_i\right\} \Bigg(1 \\
&+ \frac{\theta}{2}\log\lambda - \frac{\theta}{2}\log\left(\lambda + e^{\frac{2}{x_i}}\right) \Bigg) \\
&- \csc^2\left\{\frac{\pi}{2}\eta_i\right\} \Bigg(\frac{\pi}{2}\eta_i \Bigg(1 \\
&+ \theta\log\lambda - \theta\log\left(\lambda + e^{\frac{2}{x_i}}\right) \Bigg) \Bigg) \Bigg) \\
&- \frac{\pi^2}{4}\theta{\eta_i}^2 \Bigg(\lambda^{-1} \\
&- \left(\lambda + e^{\frac{2}{x_i}}\right)^{-1} \Bigg) \csc^2\left\{\frac{\pi}{2}\eta_i\right\} \Bigg(\log\lambda \\
&- \log\left(\lambda + e^{\frac{2}{x_i}}\right) \Bigg)^2
\end{aligned}
$$

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$$L_{122} = L_{221} = -\frac{n}{\lambda^2} + \sum_{i=1}^{n} \frac{1}{\left(\lambda + e^{\frac{2}{x_i}}\right)^2}$$

$$+ \sum_{i=1}^{n} -\csc^2\left\{\frac{\pi}{2}\eta_i\right\}\left(\frac{\pi^2}{2}\theta\eta_i^{\,2}\left(\lambda^{-1}\right.\right.$$

$$\left.-\left(\lambda + e^{\frac{2}{x_i}}\right)^{-1}\right)^2\left(1\right.$$

$$\left.\left.+\theta\log\lambda - \theta\log\left(\lambda + e^{\frac{2}{x_i}}\right)\right)\right)$$

$$-\csc^2\left\{\frac{\pi}{2}\eta_i\right\}\left(\frac{\pi^2}{4}\theta\eta_i^{\,2}\left(\theta\left(\lambda^{-1}\right.\right.\right.$$

$$\left.-\left(\lambda + e^{\frac{2}{x_i}}\right)^{-1}\right)^2$$

$$\left.-\left(\lambda^{-2} - \left(\lambda + e^{\frac{2}{x_i}}\right)^{-2}\right)\right)\left(\log\lambda\right.$$

$$\left.\left.-\log\left(\lambda + e^{\frac{2}{x_i}}\right)\right)\right)$$

$$+\cot\left\{\frac{\pi}{2}\eta_i\right\}\csc^2\left\{\frac{\pi}{2}\eta_i\right\}\left(\frac{\pi^3}{4}\theta^2\eta_i^{\,3}\left(\lambda^{-1}\right.\right.$$

$$\left.-\left(\lambda + e^{\frac{2}{x_i}}\right)^{-1}\right)^2\left(\log\lambda\right.$$

$$\left.\left.-\log\left(\lambda + e^{\frac{2}{x_i}}\right)\right)\right)$$

$$+\cot\left\{\frac{\pi}{2}\eta_i\right\}\left(\left(\pi\theta\eta_i\left(\lambda^{-1}\right.\right.\right.$$

$$\left.-\left(\lambda + e^{\frac{2}{x_i}}\right)^{-1}\right)^2\left(1\right.$$

$$\left.\left.+\frac{\theta}{2}\log\lambda - \frac{\theta}{2}\log\left(\lambda + e^{\frac{2}{x_i}}\right)\right)\right)$$

$$-\left(\frac{\pi}{2}\eta_i\left(\lambda^{-2} - \left(\lambda + e^{\frac{2}{x_i}}\right)^{-2}\right)\left(1\right.\right.$$

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$$+\theta\log\lambda-\theta\log\left(\lambda+e^{\frac{2}{x_i}}\right)\Bigg)\Bigg)\Bigg)$$

$$L_{222}=\frac{2n\theta}{\lambda^3}-(\theta+1)\sum_{i=1}^{n}\frac{2}{\left(\lambda+e^{\frac{2}{x_i}}\right)^3}$$
$$+\sum_{i=1}^{n}\cot\left\{\frac{\pi}{2}\eta_i\right\}\left(\frac{3}{2}\pi\theta^2\eta_i\left(\lambda^{-2}\left(\lambda\right.\right.\right.$$
$$\left.+e^{\frac{2}{x_i}}\right)^{-1}(1-\theta)+\lambda^{-1}\left(\lambda+e^{\frac{2}{x_i}}\right)^{-2}(1+\theta)\Bigg)$$
$$+\pi\theta\eta_i\left(\lambda^{-3}\left(1-\frac{3}{2}\theta+\frac{1}{2}\theta^2\right)\right.$$
$$\left.-\left(\lambda+e^{\frac{2}{x_i}}\right)^{-3}\left(1+\frac{3}{2}\theta+\frac{1}{2}\theta^2\right)\right)\Bigg)$$
$$-\csc^2\left\{\frac{\pi}{2}\eta_i\right\}\left(\pi^2\theta^2{\eta_i}^2\left(\frac{\theta}{2}\left(\lambda^{-1}\right.\right.\right.$$
$$\left.-\left(\lambda+e^{\frac{2}{x_i}}\right)^{-1}\right)^3$$
$$-\frac{1}{2}\left(\lambda^{-3}+\left(\lambda+e^{\frac{2}{x_i}}\right)^{-3}-\lambda^{-1}\left(\lambda+e^{\frac{2}{x_i}}\right)^{-2}\right.$$
$$\left.\left.\left.-\lambda^{-2}\left(\lambda+e^{\frac{2}{x_i}}\right)^{-1}\right)\right)\right)$$
$$-\csc^2\left\{\frac{\pi}{2}\eta_i\right\}\left(\frac{\pi^2}{4}\theta^2{\eta_i}^2\left(\left(\lambda^{-1}\right.\right.\right.$$
$$\left.-\left(\lambda+e^{\frac{2}{x_i}}\right)^{-1}\right)\left(\theta\left(\lambda^{-1}-\left(\lambda+e^{\frac{2}{x_i}}\right)^{-1}\right)^2\right.$$
$$\left.\left.\left.\left.-\left(\lambda^{-2}-\left(\lambda+e^{\frac{2}{x_i}}\right)^{-2}\right)\right)\right)\right)\right)$$
$$+\cot\left\{\frac{\pi}{2}\eta_i\right\}\csc^2\left\{\frac{\pi}{2}\eta_i\right\}\left(\frac{\pi^3}{4}\theta^3{\eta_i}^3\left(\lambda^{-1}\right.\right.$$
$$\left.\left.-\left(\lambda+e^{\frac{2}{x_i}}\right)^{-1}\right)^3\right)$$

If $u(\theta,\lambda)=\theta$, the approximate Bayes estimates of θ SELF is given by

$$\hat{\theta}_{BS}\simeq\theta+\rho_1\sigma_{11}+\rho_2\sigma_{12}+\frac{1}{2}\left(L_{111}\sigma_{11}^2+3L_{112}\sigma_{12}\sigma_{11}+L_{122}(\sigma_{22}\sigma_{11}+2\sigma_{12}^2)+L_{222}\sigma_{22}\sigma_{12}\right)$$

and similarly, the Bayes estimator for λ under SELF is:

$$\hat{\lambda}_{BS}\simeq\lambda+\rho_1\sigma_{21}+\rho_2\sigma_{22}+\frac{1}{2}\left(L_{111}\sigma_{11}\sigma_{12}+L_{112}(\sigma_{11}\sigma_{22}+2\sigma_{12}^2)+3L_{122}\sigma_{12}\sigma_{22}+L_{222}\sigma_{22}^2\right)$$

If $u(\theta,\lambda)=\log(\theta)$, the approximate Bayes estimates of θ SLLF is given by

$$\hat{\theta}_{BL}\simeq exp\left\{\log(\theta)+\frac{1}{\theta}\rho_2\sigma_{12}+\frac{1}{\theta}\left(\rho_1-\frac{1}{2\theta}\right)\sigma_{11}+\frac{1}{2\theta}\left(L_{111}\sigma_{11}^2+3L_{112}\sigma_{12}\sigma_{11}+L_{122}(\sigma_{22}\sigma_{11}+2\sigma_{12}^2)+L_{222}\sigma_{22}\sigma_{12}\right)\right\}$$

Also, similarly, the Bayes estimator of λ under SLLF is:

$$\hat{\lambda}_{BL}\simeq exp\left\{\log(\lambda)+\frac{1}{\lambda}\rho_1\sigma_{21}+\frac{1}{\lambda}\left(\rho_2-\frac{1}{2\lambda}\right)\sigma_{22}+\frac{1}{2\lambda}\left(L_{111}\sigma_{11}\sigma_{12}+L_{112}(\sigma_{11}\sigma_{22}+2\sigma_{12}^2)+3L_{122}\sigma_{12}\sigma_{22}+L_{222}\sigma_{22}^2\right)\right\}$$

Further the Bayes estimator of the reliability function under SELF and SLLF are given following.
Bayes estimator for reliability function $R(t)$
$u_R(\theta,\lambda)=R(t)$, then the corresponding derivatives are

$$u_{R1}=\frac{\partial u_R}{\partial\theta}=-\sin\{\psi\}\left\{\psi\left(\log(\lambda)-\log\left(\lambda+e^{\frac{2}{t}}\right)\right)\right\}$$

$$u_{R11}=\frac{\partial^2u_R}{\partial\theta^2}=-(\cos\{\psi\}(\psi)+\sin\{\psi\})\left\{\psi\left(\log(\lambda)-\log\left(\lambda+e^{\frac{2}{t}}\right)\right)^2\right\}$$

$$u_{R12}=\frac{\partial^2u_R}{\partial\theta\partial\lambda}=-\psi\left(\lambda^{-1}-\left(\lambda+e^{\frac{2}{t}}\right)^{-1}\right)\left\{\left(\cos\{\psi\}(\theta\psi)\left(\log(\lambda)-\log\left(\lambda+e^{\frac{2}{t}}\right)\right)\right)+\left(\sin\{\psi\}\left(1+\theta\log(\lambda)+\theta\log\left(\lambda+e^{\frac{2}{t}}\right)\right)\right)\right\}$$

$$u_{R2}=\frac{\partial u_R}{\partial\lambda}=-\sin\{\psi\}\left\{(\theta\psi)\left(\lambda^{-1}-\left(\lambda+e^{\frac{2}{t}}\right)^{-1}\right)\right\}$$

$$u_{R22}=\frac{\partial^2u_R}{\partial\lambda^2}=-\cos\{\psi\}(\theta\psi)^2\left(\lambda^{-1}-\left(\lambda+e^{\frac{2}{t}}\right)^{-1}\right)^2-\sin\{\psi\}\left\{(\theta\psi)\left(\left(\left(\lambda+e^{\frac{2}{t}}\right)^{-2}-\lambda^{-2}\right)+\theta\left(\lambda^{-1}-\left(\lambda+e^{\frac{2}{t}}\right)^{-1}\right)^2\right)\right\}$$

where $\psi=\frac{\pi}{2}\lambda^{\theta}\left(\lambda+e^{\frac{2}{t}}\right)^{-\theta}$

The Bayes estimator of reliability function under SELF is:

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$$\hat{R}(t)_{BS} \simeq \mathrm{R}(t) + \frac{1}{2}[(u_{11} + 2u_1\rho_1)\sigma_{11} + (u_{21} + 2u_2\rho_1)\sigma_{21} + (u_{12} + 2u_1\rho_2)\sigma_{12} + (u_{22} + 2u_2\rho_2)\sigma_{22}] + \frac{1}{2}[(u_1\sigma_{11} + u_2\sigma_{12})(L_{111}\sigma_{11} + L_{121}\sigma_{12} + L_{211}\sigma_{21} + L_{221}\sigma_{22}) + (u_1\sigma_{21} + u_2\sigma_{22})(L_{211}\sigma_{11} + L_{122}\sigma_{12} + L_{212}\sigma_{21} + L_{222}\sigma_{22})]$$

and, the Bayes estimator for reliability function under SLLF is:

$$\hat{R}(t)_{BL} = exp\left\{\log(R(t)) + \frac{1}{2}[(u_{11} + 2u_1\rho_1)\sigma_{11} + (u_{21} + 2u_2\rho_1)\sigma_{21} + (u_{12} + 2u_1\rho_2)\sigma_{12} + (u_{22} + 2u_2\rho_2)\sigma_{22}] + \frac{1}{2}[(u_1\sigma_{11} + u_2\sigma_{12})(L_{111}\sigma_{11} + L_{121}\sigma_{12} + L_{211}\sigma_{21} + L_{221}\sigma_{22}) + (u_1\sigma_{21} + u_2\sigma_{22})(L_{211}\sigma_{11} + L_{122}\sigma_{12} + L_{212}\sigma_{21} + L_{222}\sigma_{22})]\right\}$$

IV. Simulation Study

In this section, a Monte Carlo simulation study is carried out to compare the performance of the Bayesian estimates under squared error loss function and square logarithmic loss function. We take random samples of size $n = 150$, 250, 300 and 500. The have been generated by the inverse transformation methods from CILCR distribution with $(\theta, \lambda) = (0.2,1)$ and choice of hyper-parameters is assumed as $a = 2$, $b = 2$, $c = 2$ and $d = 2$. The results are replicated 1000 times and simulation results are summarised in Table 1. As we can see that Bayesian estimates are performance quite better than their counterparts e.g. Maximum Likelihood estimate (MLE). All results are obtained using Mathematica11.

Table 1: Average estimates and corresponding mean square error (MSE) of the parameters θ and λ and the reliability function R(t).

n	MLE		SELF		SLLF		$R(t=2) = 0.35493$	
	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{R}(t)_{BS}$	$\hat{R}(t)_{BL}$
150	0.20454 (0.00052)	1.07347 (0.10499)	0.20470 (0.00004)	1.08504 (0.01916)	0.20360 (0.00004)	1.04802 (0.06298)	0.35275 (0.00009)	0.35417 (9.81636×10^{-6})
250	0.20368 (0.00031)	1.06156 (0.06152)	0.20250 (7.38905×10^{-6})	1.04307 (0.00260)	0.20181 (4.36542×10^{-6})	1.0177 (0.00089)	0.35427 (4.88284×10^{-6})	0.35470 (6.01714×10^{-7})
300	0.20321 (0.00026)	1.05424 (0.04931)	0.20202 (4.38013×10^{-6})	1.03432 (0.00140)	0.20145 (2.35638×10^{-6})	1.01333 (0.00029)	0.35450 (1.28972×10^{-6})	0.35478 (1.61827×10^{-7})
500	0.20176 (0.00015)	1.03166 (0.02930)	0.20118 (1.45433×10^{-6})	1.01992 (0.00044)	0.20084 (7.51429×10^{-7})	1.00742 (0.00007)	0.35473 (2.28422×10^{-7})	0.35486 (2.87324×10^{-8})

V. Conclusion and future works

Bayes estimators of the unknown parameters and the reliability function for Cosine inverse log compound Rayleigh distribution have been considered. From Table 1, we have observed that the Bayes estimates tend to converge to the true parametric values by increasing the sample size. Also, the mean squared error of the Bayes estimates decreases as the sample sizes increases. Due the aim of the current research, the Bayesian estimation of CILCR parameters only discussed to simulated data. Therefore, it will be worthy if Bayesian estimation techniques to the parameters of CILCR distribution is investigated case of modelling a real dataset.

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