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Topological Spaces Associated with Finite Divisor Graphs

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Abstract— The aim of this paper is to represent a bitopological representation $(V, \tau_{S_1}, \tau_{S_2})$ of divisor graph $G = (V, E)$ defining in a finite commutative rings in which every vertex v is adjacent with a vertex u if and only if $g.c.d(u, v) = 1$. Then some properties of this bitopological space were investigated.

Keywords— *Finite Ring, Divisor Graph, degree of vertex, Bitopological space*.

1. Introduction

In (2013), S. Amiri, A. Jafarzadeh, H. Khatibzadeh present a definition of an Alexandroff topology and in (2018), K. Abdu, A. Kilicman use this topolology and defined another one to give a bitopological spaces on undirected graphs. The reader can refer to [5,7]. This paper introduces and studies a Topological Spaces Associated with Finite Divisor Graphs $G = (V, E)$. Specifically, for a finite commutative ring $R = (\mathbb{Z}_n, +_n, \cdot_n)$, where $\mathbb{Z}_n = \{0,1,2,3, \dots, n-1\}$ and $(+_n), (\cdot_n)$ are addition and multiplication module the integer n, at it has been defined in [3]. We consider the graph $G = (V, E)$ where vertices are ring elements and two vertices u and v are adjacent if and only if their greatest common divisor is 1 without accounting the loop at the vertex 1, Because we are taking into consideration $g.c.d(u, v) = 1$ if and only if $u \neq v$. On such a graph, we define two topologies τ_{S_1} and τ_{S_2} via subbases S_1 and S_2 derived from adjacency sets, leading to a bitopological space $(V, \tau_{S_1}, \tau_{S_2})$.
Our main objective is to explore the topological properties of these structures, including separation axioms, connectedness, regularity, and the behavior of open and closed sets. We also examine how graph-theoretic properties, such as vertex degree and adjacency, influence the topological features of the associated spaces.
The paper is organized as follows: Section 2 provides necessary preliminaries from graph theory and topology. Sections 3 and 4 present our main results on the topological properties of τ_{S_1} and τ_{S_2}, respectively. Section 5 is devoted to bitopological properties, and the final section offers concluding remarks and directions for future research.

2. Preliminaries

In this section, we recall basic definitions and notations from graph theory and topology that will be used throughout the paper. Standard references include [1,2,4,6, 8].

2.1 Preliminaries on graphs

A simple graph $G = (V, E)$ consists of a vertex set V and an edge set $E \subseteq V \times V$. Two vertices x and y are adjacent if $xy \in E$. The set of neighbors of a vertex v is denoted by A_v. The degree of v, denoted $deg(v)$, is the number of neighbors of v.
A divisor graph over a finite commutative ring $\mathbb{Z}_n$ is defined as $G = (V, E)$ where $V = \mathbb{Z}_n$ and $uv \in E$ if and only if $g.c.d(u, v) = 1$ for all $u \neq v$.

2.2 *Preliminaries on topology*

In this subsection, we define the topologies τ_{S_1} which has been interfused in [4] and give a new definition of an another topology τ_{S_2} on the same vertex set V of a divisor graph, leading to a bitopological space that captures both algebraic and combinatorial properties of the underlying ring. For more information, see [6,8].

Definition 2.2.1. [6] A topology on a set X is a collection τ of subsets of X having the following properties:

(1) $\emptyset$ and X are in τ .
(2) The union of the elements of any subcollection of τ is in τ.
(3) The intersection of the elements of any finite subcollection of τ is in τ.
A set X for which a topology τ has been specified is called a *topological space*.

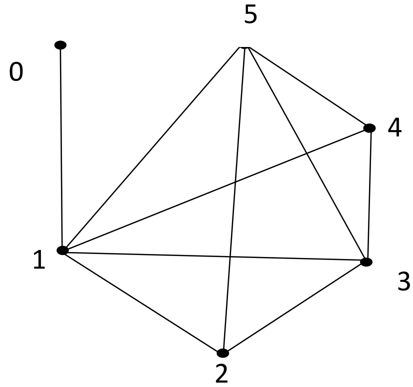
If X is a topological space with topology τ, we say that a subset U of X is an ***open set*** of X if U belongs to the collection τ. Using this terminology, one can say that a topological space is a set X together with a collection of subsets of X, called open sets, such that $\emptyset$ and X are both open, and such that arbitrary unions and finite intersections of open sets are open.

Definition 2.2.2. [6] If X is a set, a *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that
(1) For each $x \in X$, there is at least one basis element B containing x.
(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a base element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Definition 2.2.3. [6] A subbase S for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbase S is defined to be the collection τ of all unions of finite intersections of elements of S.

Definition 2.2.4. [4] The topology τ_{S_1} is defined as follows:
Let $G = (V, E)$ is a simple undirected finite graph and with no isolated vertices, that is for each $u \in V$, there exist $v \in V$ such that the edge $\{u, v\} \in E$, this means x and y are adjacent $(u \sim v)$. Let $A_v = \{u \in V ; \{u, v\} \in E\}$ be the neighborhood of v. The topology τ_{S_1} on the set V is the topology which has the collection S_1 as a subbase, where $S_1 = \{A_v : v \in V \}$. We say the pair (V, τ_{S_1}) is a graphic topological space or (V, τ_{S_1}) is topological graph.

Example 2.2.5.
Let $G = (\mathbb{Z}_6, E)$, where $E = \{uv : g.c.d(u, v) = 1 , u \neq v\}$, as show in *Figure (1)* below. Then
$S_1 = \{A_0 = \{1\}, A_2 = A_4 = \{1,3,5\}, A_5 = \{1,2,3,4\}, A_3 = \{1,2,4,5\}, A_1 = \{0,2,3,4,5\}\}$
and
$\mathcal{B}_{S_1} = \{\emptyset, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{2,4\}, \{3,5\}, \{1,2,4\}, \{1,3,5\}, \{2,3,4\}, \{2,4,5\}, \{1,2,3,4\}, \{1,2,4,5\}, \{0,2,3,4,5\}\}$.
Hence
$\tau_{S_1} = \{\emptyset, \mathbb{Z}_6, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{3,5\}, \{1,3,5\}, \{2,4\}, \{1,2,4\}, \{2,3,4\}, \{2,4,5\}, \{1,2,3,4\}, \{1,2,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}, \{0,2,3,4,5\}\}$.



$\boldsymbol{Figure}$ $(\mathbf{1})$

Now If we suppose that $O_v = A_v \cup \{v\}$. Then we can investigate an another topology τ_{S_2} on the set of vertices V, which defined by the $subbasis\ elements\ O_{v_i}$ as following:

Definition 2.2.6.
A topological space (V, τ_{S_2}) where $V = \mathbb{Z}_n$ and τ_{S_2} is a topology defined by the subbase $S_2 = \{O_v : v \in V\}$.

Example 2.2.7.
Let $G = (\mathbb{Z}_6, E)$, The same graph which is defined in *Figure (1)*. Then
$S_2 = \{O_0 = \{0,1\}, O_1 = \mathbb{Z}_6, O_2 = \{1,2,3,5\}, O_3 = O_5 = \{1,2,3,4,5\}, O_4 = \{1,3,4,5\}\}$ and
$\mathcal{B}_{S_2} = \{\emptyset, \mathbb{Z}_6, \{1\}, \{0,1\}, \{1,3,5\}, \{1,2,3,5\}, \{1,3,4,5\}, \{1,2,3,4,5\}\}$.
Hence
$\tau_{S_2} = \{\emptyset, \{\mathbb{Z}_6, \{1\}, \{0,1\}, \{1,3,5\}, \{0,1,3,5\}, \{1,2,3,5\}, \{1,3,4,5\}, \{0,1,2,3,5\}, \{0,1,3,4,5\}, \{1,2,3,4,5\}\}$.

Definition 2.2.8.
A bitopological space is a triple $(V, \tau_{S_1}, \tau_{S_2})$ where τ_{S_1} and $\tau_{S_2}$are topologies on $V = \mathbb{Z}_n$.

3. Main Results

In this section, investigate the properties of the topological spaces (V, τ_{S_1}) and (V, τ_{S_2}), where $V = \mathbb{Z}_n$ and $n \geq 4$. Moreover we study some of properties of the bitopological space $(V, \tau_{S_1}, \tau_{S_2})$.

3.1 The topological space $(\boldsymbol{V}, \boldsymbol{\tau_{S_1}})$

In this section, we investigate properties of the topological space (V, τ_{S_1}), where $V = \mathbb{Z}_n$ and $n \geq 4$.

Proposition 3.1.1 Let $G = (V, E)$ be a divisor graph. Then in the topological space (V, τ_{S_1}), the set $\{0\}$ is closed but not open.

Proof. To prove that $\{0\}$ is closed set, it is enough to prove that $(\{0\})^c$ can be written as an arbitrary union of members of S_G. But for any $1 \neq v \in V$ we have $0 \notin A_v$ that means $0 \notin \bigcup_{v \neq 1} A_v = V \setminus \{0\} \in \tau$, so $\{0\}$ is closed set. On the other hand, the only open set containing the vertex 0 in S_G is $A_1 = V \setminus \{1\}$ which means that $\{0\}$ can-not be obtained from any finite intersection of elements in S_G. □

Proposition 3.1.2. Let $G = (V, E)$ be a divisor graph. Then in the topological space (V, τ_{S_1}), the set $\{1\}$ is clopen.

Proof. Since $g.c.d(0, v) = 1$ if and only if $v = 1$, then $A_0 = \{1\}$ is an open set. On the other hand, $g.c.d(1, v) = 1$ for all $v \in V$ which means $A_1 = V \setminus \{1\} \in \tau$, so the set $\{1\}$ is closed. □

Remark 3.1.3. Proposition 3.1.2 shows that the only vertex in the divisor graph $G = (V, E)$ with degree $n - 1$ is the vertex 1, as established in its proof**.**

Proposition 3.1.4. Let $G = (V, E)$ be a divisor graph. Then in the topological space (V, τ_{S_1}), if $deg(v) = n - 2$, the set $A = \{v\}$ is an open.
Then for any $v \in V$, if $deg(v) = n - 2$, then the singleton $\{v\}$ is open in (V, τ_{S_1}).

Proof. A vertex v has degree $\text{n} - 2$ if and only if it is adjacent to all vertices $w_i \in V \setminus \{0, v\}$ if and only if $g.c.d(v, w_i) = 1$, so $v \in A_{w_i}$ for all i which means $v \in \bigcap_{w_i \in V \setminus \{0,v\}} A_{w_i}$. But $w_i \notin A_{w_i}$ for any i, which leads us to the conclusion that $w_i \notin \bigcap_{w_i \in V \setminus \{0,v\}} A_{w_i}$. Hence the vertex v is the unique element in the intersection of all these sets. Therefore $\{v\} = \bigcap_{w_i \in V \setminus \{0,v\}} A_{w_i}$, so $\{v\}$ is a finite intersection of sub basis elements ⇒ open. □

Proposition 3.1.5. Let $G = (V, E)$ be a divisor graph. Then for any integer $k \geq 1$ with $2 \leq 2^k < n$, we have $A_2 = A_{2^K}$.
Proof. Let $v \in V$. A vertex $v \in A_2$ if and only if $g.c.d(2, v) = 1$ if and only if $g.c.d(2^k, v) = 1$, so $v \in A_{2^K}$. Therefor $A_2 = A_{2^K}$. □

Corollary 3.1.6. Let $G = (V, E)$ be a divisor graph with $V = \mathbb{Z}_n$ for $n > 4$, then:

1- For vertex $v = 2^K$ such that, the integer $k \geq 1$ with $2 \leq 2^k < n$. The set $\{v\}$ is not closed in the topological space (V, τ_{S_1}).

2- The topological space (V, τ_{S_1}) is not T_0.

Proof:

1- Immediately from Proposition 3.1.5. Since $|A_2| \geq 2$. □

2- Immediately from Proposition 3.1.5. Let $v_1 = 2$ and $v_2 = 4$. □

Remark 3.1.7. Proposition 6.2 in [7] does not hold in our framework because of the pendent vertex 0. As an example, in $(\mathbb{Z}_5, \tau_{S_1})$ the set $\{2, 4\} \in \tau$. But $(\{2, 4\})^c \notin \tau$.

Proposition 3.1.8. Let $G = (V, E)$ be a divisor graph. Then in the topological space (V, τ_{S_1}), the only open sets containing the vertex 0 are $V = \mathbb{Z}_n$ and A_1.

Proof. Since $g.c.d(0,1) = 1$ and for any another vertex $v \neq 1$ we have $g.c.d(0, v) \neq 1$, so $0 \in A_1$, $0 \notin A_v$, and $1 \in A_v$ because $g.c.d(1, v) = 1$. Hence $A_1 \cup A_v = \begin{cases} A_1 & if\ v = 1 \\ V & if\ v \neq 1 \end{cases}$. □

Proposition 3.1.9. Let $G = (V, E)$ be a divisor graph. Then (V, τ_{S_1}) is a disconnected topological space.

Proof. Let $U = \{1\}$ and $W = U^c$. Then by Proposition 3.1.2., we have $U, W \in \tau$, $U \cup W = V$, and $U \cap W = \emptyset$. □

3.2 The topology (V, τ_{S_2})

In this section, we investigate properties of the topological space (V, τ_{S_2}), where $V = \mathbb{Z}_n$ and $n \geq 4$.

Proposition 3.2.1. Let $G = (V, E)$ be a divisor graph. Then in the topological space (V, τ_{S_2}), the set $\{1\}$ is an open but not closed.

Proof. Proof. Since $g.c.d(0, v) = 1$ if and only if $v = 1$, then $O_0 = \{0,1\}$ and $0 \notin O_v$ for any $v \neq 1$, so $\{1\} = O_0 \cap O_v$ is an open set. On the other hand, $g.c.d(1, v) = 1$ for all $v \in V$ that means every nonempty proper open set U must contains the vertex 1 , so $1 \notin U^c$. Hence, the set $\{1\}$ can-not be closed. □

Proposition 3.2.2. Let $G = (V, E)$ be a divisor graph, and let $v \in V$ with $v \neq 1$. Then the singleton $F = \{v\}$ is not open in the topological space (V, τ_{S_2}).

Proof. Since $g.c.d(1, v) = 1$ for every $v \in V$, the vertex 1 is adjacent to all other vertices in G. Consequently, 1 belongs to the interior F° of F, so $F^{\circ} \neq F$. Therefore F is not open set. □

Corollary 3.2.3. Let $G = (V, E)$ be a divisor graph. Then in the topological space (V, τ_{S_2}), every non-empty open set must contain the element 1.

Proposition 3.2.4. Let $G = (V, E)$ be a divisor graph. Then (V, τ_{S_2}) is a connected topological space.

Proof. Immediately from Corollary 3.2.3.

Proposition 3.2.5. Let $G = (V, E)$ be a divisor graph. Then the topological space (V, τ_{S_2}) is not regular.

Proof. By Corollary 3.2.3. If B is a proper closed set of V, then $1 \notin B$. However, every open set containing B must contain the vertex 1. Thus the vertex 1 and the closed set B cannot be separated by disjoint open sets, violating the definition of a regular topological space. □

Proposition 3.2.6. Let $G = (V, E)$ be a divisor graph. Then in the topological space (V, τ_{S_2}), the set $\{0\}$ is closed.

Proof. Let $v \neq 2$ be a prime vertex such that $mv > n$ for any positive integer $1 < m < n$. Then by (Theorem 6) in [3], $\deg(v) = n - 1$ so a vertex v is adjacent to all vertices except 0, which means $O_v = X \setminus \{0\} \in \tau$. Hence $\{0\}$ is closed set. □

Proposition 3.2.7. Let $G = (V, E)$ be a divisor graph with $V = \mathbb{Z}_n$ for $n > 4$ and suppose that $v = 2^K$ where $k \geq 1$ and $2 \leq 2^k < n$. Then the singleton $\{v\}$ is a closed set in the topological space (V, τ_{S_2}).

Proof. From the Proposition 3.1.5, we have $O_2 \setminus \{2\} = A_{2^K} \setminus \{2^k\}$. Now if we take all even vertices $u_i \neq 2^k$ with $1 \leq u_i < n - 1$, then from the definition of adjacency in the divisor graph, u_i are adjacent to vertices $u_i - 1, u_i + 1$ but not adjacent to the vertex 2^k, so $(\cup_{\forall_i} A_{u_i}) = V \setminus \{2^k\}$. This shows that the complement of $\{2^k\}$ is open in τ_S, and therefore $\{2^k\}$ is closed. □

Theorem 3.2.8. The topological space (V, τ_{S_2}) is not a T_0 space.

Proof. Let u and v be distinct prime vertices different from 2 such that $mu, rv \geq n$ for all integers m with $1 < m < n$. By Theorem 6 in [3], this condition implies that both u and v are adjacent to every vertex except 0.
Therefore, for any open set U in τ_G other than $O_0 = \{0,1\}$, u and v belong to U. Hence (V, τ_{S_2}) fails to satisfy the T_0 separation axiom. □

4- Bitopological spaces on divisor graphs

In the following we give some properties of a bitopological space $(V, \tau_{S_1}, \tau_{S_2})$ of a divisor graph $G = (V, E)$ which is defined in a finite commutative ring $(\mathbb{Z}_n, +_n, \cdot_n)$.

Proposition 4.1. In the bitopological space $(V, \tau_{S_1}, \tau_{S_2})$ the sets $\{1\}$ and $V \setminus \{0\}$ are open.

Proof. If $U_1 = \{1\}$, then by Propositions (3.1.2), and (3.2.1) we find that U_1 is an open set in the bitopological space $(V, \tau_{S_1}, \tau_{S_2})$.
Now to prove that $U_2 = V \setminus \{0\}$ is an open set, one can choose any prime vertex $v \neq 2$ such that $mv > n$ for any positive integer $m < n$. Then by (Theorem 6) in [3], $deg(v) = n - 2$, so $A_v = V \setminus \{0, v\}$ and since $v \in A_{v-1}$ which means $U_2 = A_v \cup A_{v-1} \in \tau_{S_1}$. On the other hand $U_v = O_v \in \tau_{S_2}$. Hence U_2 is an open set in the bitopological space $(V, \tau_{S_1}, \tau_{S_2})$. □

Proposition 4.2. The set $\{0\}$ is closed in the bitopological space $(V, \tau_{S_1}, \tau_{S_2})$.

Proof. See Propositions (3.1.1) and (3.2.6). □

Proposition 4.3 The bitopological space $(V, \tau_{S_1}, \tau_{S_2})$ is not T_0.

Proof. See Corollary 3.1.6 and Theorem (3.2.8). □

Proposition 4.4 The bitopological space $(V, \tau_{S_1}, \tau_{S_2})$ is not regular.

Proof. See Propositions (3.2.5). □

5- Conclusion

In this paper, we have introduced and studied a bitopological space $(V, \tau_{S_1}, \tau_{S_2})$ associated with divisor graphs defined over finite commutative rings $\mathbb{Z}_n$. We investigated fundamental topological properties such as connectedness, separation axioms, and the nature of open and closed sets in both topologies.

Key findings include:

- The space (V, τ_{S_1}) is disconnected and not T_0, with distinguished behavior of vertices such as 0 and 1.
- The space (V, τ_{S_2}) is connected but not regular or T_0, with vertex 1 playing a central role in every nonempty open set.
- The bitopological space $(V, \tau_{S_1}, \tau_{S_2})$ neither regular nor T_0. Moreover, we have proved that sets such as $\{1\}$ and $V\setminus\{0\}$ are open, while $\{0\}$ is closed.

6- Further work

These results illustrate how graphic properties of finite rings influence its topological structure. Future work may extend this approach by using (Definition 2.2.6) to make a topology representation for any graph G whether it was finite or infinite, and connected or disconnected.

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