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Advanced Study of Soft Sets and Soft Matrices: Theory, Properties, and Application

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***Abstract*— Soft set theory, introduced by Molodtsov in 1999, addresses uncertainty and imprecision. This paper proposes enhancements to soft set theory, including extended relative and h-dependent complements, addressing decision-making under uncertainty. Matrix representations are also refined for better computational utility.**

***Keywords*—Soft sets, Soft matrices, Complement operations, Intersection refinements, Decision-making applications, Mathematical modeling.**

1. Introduction

Addressing uncertainty, imprecision, and incomplete information is a key challenge in mathematics, logic, and computer science. This has led to the creation of models that handle ambiguity, such as Fuzzy Sets [1], Rough Sets [2], and Soft Sets [3], each offering unique approaches to managing uncertainty, with varying degrees of flexibility for complex data.

Fuzzy and rough set theories laid the groundwork for dealing with vagueness and approximation. However, soft set theory, introduced by Molodtsov in 1999, provides greater flexibility through its parameterized structure, making it especially suitable for decision-making under incomplete data. Subsequent work significantly expanded the theory. Studies [4-6]. introduced and refined algebraic operations like union, intersection, and complement, resolving inconsistencies and extending functionality. Applications in decision-making were further developed through parameter reduction and uncertainty-handling technique [7-10]. Moreover, soft set theory has been unified with other models, with fuzzy and rough sets treated as special cases [11], and extended into areas such as fuzzy soft sets [12] and soft graph theory [13], broadening its impact in optimization and network analysis.

In This study introduces key enhancements to soft set theory, including the extended relative complement for broader parameter analysis and the h-dependent complement, which incorporates historical parameter influence for more informed decision-making. We also propose a standardized symbolic notation to unify inconsistent definitions and improve clarity, and we enhance matrix representations to support efficient computational applications. These contributions strengthen the theoretical framework and expand the applicability of soft sets in areas such as data analysis, decision-making, and optimization.

2. Basic Definitions.

In this section, we give some basic definitions for soft sets, throughout this paper, U denotes an initial universe set and E is a set of parameters; the power set of U is denoted by $P(U)$, and A is a subset of E . Soft sets are defined as a pair (F, A), where:

i. $F: A \to P(U)$ with $A \subset E$ representing the set of parameters and $P(U)$ being the power set of U (the universe).
ii. $F(e)$ is a subset of U for each $e \in A$, which may represent attributes or properties of elements in U see ([3]).

A soft set can be expressed as a set ordered pairs $\big(e, F(e)\big)$, where $F(e)$ is a subset of U. This representation is useful for formal analysis but can become cumbersome with large sets.

2.1 Definitions of Special Soft Sets

Definition 2.1.1 [4]: A soft set (F, A) over U is said to be null soft set, if $\forall e \in A, F(e) = \emptyset$, denoted by $\widetilde{\Phi}$.

Definition 2.1.2 [4]: A soft set (F, A) over U is said to be absolute soft set, if $\forall e \in A, F(e) = U$, denoted by $\tilde{A}$.

Definition 2.1.3 [6]: A soft set (F, A) over U is said to be relative null soft set with respect to the parameter set A, if $\forall e \in A, F(e) = \emptyset$, denoted by $\widetilde{\Phi}_A$. The relative Null soft set with respect to the set of parameters E is called the Null soft set over U and simply denoted by $\widetilde{\Phi}_E$.

Definition 2.1.4 [6]: A soft set (F, A) over U is said to be relative whole soft set with respect to the parameter set A, if $\forall e \in A, F(e) = U$, denoted by $\widetilde{U}_A$. The relative whole soft set $\widetilde{U}_E$ with respect to the universe set of parameters E is called the absolute soft set over U.

2.2 *Comparison of the Definitions*

2.2. 1 Equivalence Between Concepts

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a) The empty soft set $\widetilde{\Phi}$ and the relative empty soft set $\widetilde{\Phi}_A$ are considered equivalent.
b) Similarly, the absolute soft set $\widetilde{A}$ and the relative absolute soft set $\widetilde{U}_A$ are equivalent.

2.2. 2 Notation Ambiguity

a) The symbols $\widetilde{\Phi}_A$ and $\widetilde{U}_A$ explicitly specify the parameter set A, making the notation clearer. However, Maji et al. use the symbols $\widetilde{\Phi}$ and $\widetilde{A}$ without specifying the parameter set, leading to ambiguity.
b) The ambiguity becomes especially problematic when comparing soft sets over different parameter sets, such as when $A \neq B$.

2.2. 3 Proposed Resolutions

Standardizing Notation: A uniform notation that always includes the parameter set (e.g., $\widetilde{\Phi}_A$ and $\widetilde{U}_A$) would reduce ambiguity. To address ambiguity in soft set theory, we propose revising the definition of empty soft sets to focus solely on their function values, allowing sets over different parameter sets to be considered equal if their functions are identical. Additionally, we recommend clarifying the criteria for soft set equality whether it depends solely on function values or includes parameter sets to ensure consistency in theoretical interpretation and application

2.2. 4 Maji's Soft Subset Definition

Maji defines (F, A) as a soft subset of (G, B), if $A \subseteq B$ and for every $e \in A$, $F(e)$ and $G(e)$ are identical approximations. However, this doesn't always hold, leading to the introduction of alternative definitions by Pei and Miao and Zhu and Wen.

2.2. 5 Modifications to "Not Set" Definition

Fu Li raised concerns about how the "Not Set" $(\neg E)$ operation behaves. The original definition of the "not set" operator differs from classical set theory's complement operator, leading to the suggestion of modifications to align it with De Morgan's laws.

2.2. 6 Suggestions for Improvement

To improve clarity and consistency in soft set theory, we advocate for the standardization of notation, particularly when comparing soft sets with differing parameter sets. A revised definition of soft subsets is also necessary to address cases where function values are identical but parameter sets vary. Furthermore, the application of De Morgan's laws and the definition of the "Not Set" operation require careful review to prevent misinterpretation and ensure alignment with classical set theory principles.

3. Modifications and Contributions in Soft Set Theory

3.1 Comparison and Clarification

a) *Reevaluation of "Not Set" in Soft Set Theory:*

i. Fu Li assumed the "not set" operator behaves like classical set theory complements, but Singh and Onyeozili [14] clarified that it operates at the parameter level, not on entire sets.
ii. This distinction invalidates the direct application of De Morgan's laws in soft set theory.

b) Definition of Complement and Relative Complement in Soft Set Theory:

i. Complement $(F, A)^c$: Defined using negated parameters $\neg A$.
ii. Relative Complement $(F, A)^r$: Applies to the given parameter set A without alteration.

Definition 3.1.1: Let U be universe sets and E set of parameters with respect to U. Let (F, A) be soft sets over U and $A \subset E$. Extended relative complement of a soft set (F, A) is denoted by $(F, A)^{r_E}$ and is defined by

$(F, A)^{r_E} = (F^{r_E}, E)$ where $F^{r_E}: E \to P(U)$ is a mapping assigned as

$$F^{r_E}(\alpha) = \begin{cases} U - F(\alpha), & if\ \alpha \in A \\ U, & if\ \ \alpha \notin A \end{cases}.$$

Comparison Between Relative Complement and Extended Relative Complement

i. Relative complement applies only to the parameters set A.
ii. Extended relative complement applies to the entire parameters set E, allowing consideration of parameters beyond A.

3.2 Example and Application (In the medical case)

Example 1: Suppose a hospital wants to classify patients who are likely to have heart disease to prioritize tests and treatments. The complete set of parameters $E = \{e_1, e_2, e_3, e_4, e_5\}$ where:

$e_1 =$ Blood test results (such as cholesterol level).
$e_2 =$ Blood pressure measurement.
$e_3 =$ Evaluation of clinical symptoms (such as chest pain, shortness of breath).
$e_4 =$ Medical history (previous illnesses or genetic factors).
$e_5 =$ Electrocardiogram (ECG) results.

Accurate data may be available for parameters e_1, e_2 and e_3, while data for parameters e_4 and e_5 may be incomplete or unreliable at the present stage due to time constraints or equipment limitations. Let us assume that there are 6 patients under consideration, i.e., $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and let a set A be currently available parameters e_1, e_2 and e_3 i.e., $A = \{e_1, e_2, e_3\}$. After the study, the soft set (F, A) is formed،

$$(\boldsymbol{F}, \boldsymbol{A}) = \{(e_1, \{h_1, h_2, h_5\}), (e_2, \{h_3, h_4\}), (e_3, \{h_4, h_5\})\}$$

Calculate the Extended relative complement:

$$(F, A)^{r_E} = \begin{cases} U - F(\alpha), & \text{if } \alpha \in A \\ U, & if\ \ \alpha \notin A \end{cases}$$

$$(F, A)^{r_E} = \{(e_1, \{h_3, h_4, h_6\}), (e_2, \{h_1, h_2, h_5, h_6\}), (e_3, \{h_1, h_2, h_3, h_6\}), (e_4, U), (e_5, U)\}.$$

Here, parameters e_4 and e_5 (history and ECG findings) are considered neutral due to lack of information or unreliability at the current stage.

3.3 The h-Dependent Complement

To overcome certain limitations in the classical definitions of soft set complement operations, this study proposes a new

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formulation referred to as the h-dependent complement. The proposed definition takes into account the historical presence of elements within the approximate functions of parameters not included in the current parameter set. By integrating this historical influence, the h-dependent complement preserves the traditional behavior for parameters within the set, while dynamically excluding elements that have appeared elsewhere, thereby refining the decision-making process and enhancing representational accuracy.

Definition 3.3.1 (**h-Dependent Complement**):

For a soft set (F, A) over a universal set U, the h-Dependent Complement is defined as:

$$(F, A)^{r_h} = (F^{r_h}, E)$$

where the new function F^{r_h} is given by:

a) For $\alpha \in A$:

$$F^{r_h}(\alpha) = U - F(\alpha)$$

This follows the classical complement rule.

b) For $\alpha \notin A$:

$$F^{r_h}(\alpha) = U - \bigcup_{\beta \in A} F(\beta)$$

This means that all elements that appeared in any function $F(\beta)$ where $\beta \in A$ are removed from U.

Example 2 (Verification of the h-Dependent Complement)

Let's consider a different example to verify whether the h-Dependent Complement works as intended.

first, Define the Universal Set and Parameters

- Universal Set: $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$
- Parameter Set: $E = \{e_1, e_2, e_3, e_4, e_5\}$
- Subset of Parameters: $A = \{e_1, e_2\}$.

next, we construct the Soft Set (F, A) as follows:

$$F(e_1) = \{h_1, h_2, h_3\}, F(e_2) = \{h_3, h_4\}.$$

Then, we apply the definition of the h-Dependent Complement:

1. For $e_1 \in A$, use classical complement: $F^{r_h}(e_1) = U - F(e_1) = \{h_4, h_5, h_6, h_7\}$
2. For $e_2 \in A$, use classical complement: $F^{r_h}(e_2) = U - F(e_2) = \{h_1, h_2, h_5, h_6, h_7\}$
3. For $e_3, e_4, e_5 \notin A$, use extended exclusion rule:

a) First, compute the union of all elements from parameters in A:

$$F(e_1) \cup F(e_2) = \{h_1, h_2, h_3, h_4\}$$

b) Then, apply the rule for e_3, e_4, e_5:

$$F^{r_h}(e_3) = F^{r_h}(e_4) = F^{r_h}(e_5) = U - \{h_1, h_2, h_3, h_4\} = \{h_5, h_6, h_7\}$$

Final, h-Dependent Complement Set

$$(F, A)^{r_h} = \begin{cases} (e_1, \{h_4, h_5, h_6, h_7\}) \\ (e_2, \{h_1, h_2, h_5, h_6, h_7\}) \\ (e_3, \{h_5, h_6, h_7\}) \\ (e_4, \{h_5, h_6, h_7\}) \\ (e_5, \{h_5, h_6, h_7\}). \end{cases}$$

This example demonstrates that the h-Dependent Complement is valid because it correctly excludes elements based on whether they appeared in previously analyzed parameter sets. Specifically, the behaviour observed aligns with the theoretical expectations, as outlined below:

a) the h-dependent complement preserves the classical complement behavior for parameters in

b) It differs from traditional complements by ensuring that all elements previously studied are removed, even when analyzing new parameters.

c) It is particularly useful in decision-making systems where historical data should influence classification.

4. Soft Set Operations

Various operations in soft set theory, analogous to classical set theory operations such as union, intersection, complement, and difference, will be introduced and explained.

4.1 Basic Operations

Definition 4.1.1 [4]: Union of Two Soft Sets.

The union of two soft sets (F, A) and (G, B) over the common universe U is a soft set (H, C) where $C = A \cup B$ and the mapping $H: C \to P(U)$ is defined as follows:

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ G(e) & if\ e \in B - A \\ F(e) \cup G(e) & if\ e \in A \cap B \end{cases}$$

We write $(F, A)\ \tilde{\cup}\ (G, B) = (H, C)$.

Definition 4.1.2 [4]: Intersection of Two Soft Sets.

The intersection of two soft sets (F, A) and (G, B) over the common universe U is a soft set (H, C) where $C = A \cap B$ and for each $e \in C, H(e) = F(\mathrm{e})$ or $G(\mathrm{e})$ (as both are same set). We write $(F, A)\ \tilde{\cap}_M\ (G, B) = (H, C)$. where $\tilde{\cap}_M$ represents the intersection as defined by Maji.

Concerns Raised by Pei and Miao [5]

Pei and Miao reviewed Definition 4.1.2 and raised concerns about the "matching condition" in the definition. Specifically, they proposed that the condition: $H(e) = F(e)$ or $G(e)$, should be replaced with: $H(e) = F(e) \cap G(e),\ \forall e \in A \cap B$. This modification ensures that the resulting mapping $H(e)$ explicitly reflects the intersection of the sets $F(e)$ and $G(e)$ rather than assuming they are identical.

Ahmad and Kharal's Modification [15]

Ahmad and Kharal further refined the definition of the intersection operation by introducing an additional condition:

a) The intersection of the parameter sets $A \cap B$ must be non-empty $(A \cap B \neq \phi)$. They also defined the result when $A \cap B = \phi$:

b) In this case, the intersection of the two soft sets is the null soft set, denoted as $\tilde{\phi}_\Phi$.

Definition 4.1.3 [5, 15]: Modified Intersection of Two Soft Sets The modified definition of the intersection of two soft sets (F, A) and (G, B) is as follows:

a) $C = A \cap B \neq \phi$ (the parameter sets have a non-empty intersection),
For each $e \in C$:
$H(e) = F(e) \cap G(e)$.

b) If $A \cap B = \phi$ then:

$$(F, A) \cap (G, B) = \tilde{\phi}_\Phi.$$

4.1.1 Comparison of Notations

To distinguish between the two versions of the intersection operation:

a) Maji's Intersection (Definition 4.1.2):

i. Symbol: $\widetilde{\cap}_M$,

ii. Assumes $F(e)$ and $G(e)$ are identical for $e \in A \cap B$.

b) Modified Intersection (Definition 4.1.3, Pei and Miao, Ahmad and Kharal):

i. Symbol: $\widetilde{\sqcap}$,

ii. Explicitly computes the set intersection $F(e) \cap G(e)$ for $e \in A \cap B$,

iii. Handles the case when $A \cap B = \phi$ by assigning a null soft set result.

Definition 4.1.4 [9]: Given two soft sets (F, E) and (G, E) over the same universe U then, difference of (F, E) and (G, E), denoted by $(F, E)\widetilde{-}(G, E) = (H, E)$, where $H(e) = F(e) - G(e), \forall e \in E$. Hence if the soft sets have different parameter sets, i.e., (F, A) and (G, B) two soft sets over the common universe U and a tow approximate function F and G defined from E to $P(U)$, then $(F, A)\widetilde{-}(G, B) = (H, A)$where, $H(e) = F(e) - G(e), \forall e \in A$. The function $H(e)$ is further refined based on whether the parameter e belongs to the intersection $A \cap B$:

$$H(e) = \begin{cases} F(e) - G(e), & if\ e \in A \cap B \\ F(e), & otherwise. \end{cases}$$

Zhu and Wen extend the definition to handle cases regardless of whether $A = B$ or if the approximate functions F and G are defined on the same domain E [16]: $(F, A)\widetilde{-}(G, B) = (H, C)$where, $C = A - \{e \in A \cap B: F(e) \subseteq G(e)\}$, and for all $e \in C$

$$H(e) = \begin{cases} F(e) - G(e) & if\ e \in A \cap B \\ F(e) & otherwise. \end{cases}$$

Example 3: let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be universal set, and let $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ denote the set of all parameters under consideration. Consider two parameter subset$A = \{e_2, e_3\}$ and B= $\{e_2, e_3, e_4\}$. We define three soft set (F, A),(G, A) and (K, B) as follows:

$$(F, A) = \{(e_2, \{h_1, h_2, h_5\}), (e_3, \{h_3, h_5\})\},$$
$$(G, A) = \{(e_2, \{h_3, h_5\}), (e_3, \{h_4\})\} \text{ and}$$
$$(K, B) = \{(e_2, \{h_1, h_2, h_3, h_5\}), (e_3, \{h_3\}), (e_4, \{h_4\})\}.$$

Then:

$$(F, A)\widetilde{-}(G, A) = \{(e_2, \{h_1, h_2\}), (e_3, \{h_3, h_5\}).$$
$$(F, A)\widetilde{-}(K, B) = \{\{(e_3, \{h_5\})\} \text{ and}$$
$$(K, B)\widetilde{-}(F, A) = \{(e_2, \{h_3\}), (e_4, \{h_4\})\}.$$

4.2 *Extended Operations on Soft Sets*

Definition 4.2.1 [6]: Let (F, A) and (G, B) be two soft sets defined over a universe U with parameter sets A and B, respectively. The extended operation $\widetilde{*}_\varepsilon$ is defined as:

$$(F, A)\ \widetilde{*}_\varepsilon(G, B) = (H, C),$$

Where $C = A \cup B$ and $H: C \to P(U)$ defined as: $\forall\ e \in C$

$$H(e) = \begin{cases} F(e), & if\ e \in A - B, \\ G(e). & if\ e \in B - A, \\ F(e) * G(e), & if\ e \in A \cap B. \end{cases}$$

The extended operation $\widetilde{*}_\varepsilon$ can be specialized or classified based on the specific mathematical operation or rule used when combining $F(e)$ and $G(e)$ in the case $e \in A \cap B$. Common classifications include:

1. Extended intersection [6]: The extended intersection of two soft sets (F, A) and (G, B) written as $(F, A)\ \widetilde{\sqcap}_\varepsilon\ (G, B) = (H, C)$, where

$$H(e) = \begin{cases} F(e), & if\ e \in A - B, \\ G(e), & if\ e \in B - A, \\ F(e) \cap G(e), & if\ e \in A \cap B. \end{cases}$$

a) Preserving Information from Both Sets

i. Unlike traditional intersections that focus only on shared parameters ($A \cap B$), the extended intersection retains elements from both A and B.

ii. If a parameter exists only in one of the sets, its corresponding function value is preserved.

b) Handling Differences in Parameter Sets

i. If a parameter e belongs to only one of the two parameter sets, the resulting function $H(e)$ is simply inherited from the respective set.

ii. This contrasts with classical intersections where such elements would be discarded.

c) Comparison with Classical Intersections in soft sets

i. Traditional Soft Set Intersection (Maji, Pei & Miao)

- The intersection defined by Maji, Pei, and Miao considers only the common parameters ($A \cap B$).
- This means elements that belong only to A or B are ignored, leading to potential loss of information.

ii. Extended Intersection

- The extended intersection is more inclusive, retaining parameter values from both sets while still computing intersections for shared elements.
- Unlike the traditional approach, this method is less restrictive and captures the full context of the two soft sets.

2. Extended union [6]: the extended union of two soft sets (F, A) and (G, B) written as $(F, A)\ \widetilde{\cup}_\varepsilon$ (G,B) $= (H, C)$, where

$$H(e) = \begin{cases} F(e), & if\ e \in A - B, \\ G(e), & if\ e \in B - A, \\ F(e) \cup G(e), & if\ e \in A \cap B. \end{cases}$$

3. Extended difference [17]: the extended difference of two soft sets (F, A) and (G, B) written as $(F, A)\widetilde{-}_\varepsilon(G, B) = (H, C)$, where

$$H(c) = \begin{cases} F(e), & if\ e \in A - B, \\ G(e), & if\ e \in B - A, \\ F(e) - G(e), & if\ e \in A \cap B. \end{cases}$$

Properties:

a) Asymmetry: The extended difference is not symmetric, meaning: $(F, A)\widetilde{-}_\varepsilon(G, B) \neq (G, B)\widetilde{-}_\varepsilon(F, A)$.

b) Non-Negativity: The extended difference always produces a soft set with parameter values contained in (F, A). However, it does not necessarily include only the parameter values from (F, A).

c) Empty Set Condition:

i. If $(F, A) = (G, B)$, then $(F, A)\widetilde{-}_\varepsilon(G, B)$ is an empty soft set. That's because $(F, A)\widetilde{-}_\varepsilon(G, B)$

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$$= \begin{cases} F(e), & if\ e \in A - B = \emptyset \\ G(e), & if\ e \in B - A = \emptyset \\ F(e) - G(e) = \emptyset & if\ e \in A \cap B = A\ or\ B. \end{cases}$$

ii. If $(F,A)\ \widetilde{\subseteq}\ (G,B)$, then $(F,A) \tilde{-}_{\varepsilon} (G,B)$ does not necessarily is an empty soft set for example if (F,A) and (G,B) are soft sets under the same universal U and A and $B \subseteq E$, such that

$(F,A) = \{\{(e_2, \{h_1, h_2\}), (e_6, \{h_3\})\}$,

$(G,B) = \{(e_2, \{h_1, h_2, h_5\}), (e_3, \{h_3, h_4\}), (e_6, \{h_3\})\}$,

Then, $(F,A)\ \widetilde{\subseteq}\ (G,B)$, but $(F,A) \tilde{-}_{\varepsilon} (G,B) = \{(e_3, \{h_3, h_4\})\} \neq \emptyset$.

4. Extended symmetric difference [19]: the extended symmetric difference of two soft sets (F,A) and (G,B) written as $(F,A)\ \tilde{\Delta}_{\varepsilon}\ (G,B)$, and is defined as $(F,A)\ \tilde{\Delta}_{\varepsilon}\ (G,B) = (H,C)$, where

$$H(c) = \begin{cases} F(e), & if\ e \in A - B, \\ G(e), & if\ e \in B - A. \\ F(e)\Delta G(e), & if\ e \in A \cap B. \end{cases}$$

where $F(e)\Delta G(e) = \big(F(e) - G(e)\big) \cup \big(G(e) - F(e)\big)$ represents the symmetric difference of the sets $F(e)$ and $G(e)$.

4.3 Restricted Operations on Soft Sets

Definition 4.3.1 [6]: Let (F,A) and (G,B) be two soft sets defined over a universe U with corresponding parameter sets E.The general restricted operation $\tilde{\Diamond}_R$, is defined by, $(F,A)\tilde{\Diamond}_{\Re}$ (G,B) $= (H,C)$ where $C = A \cap B \neq \emptyset$ and for each $e \in C$, $H(e) = F(e) \Diamond G(e)$. If $A \cap B = \emptyset$ then $(F,A)\ \tilde{\Diamond}_{\Re}\ (G,B) = \tilde{\Phi}_{\Phi}$.

Now the restricted operation can be classified as follows: for two soft sets (F,A) and (G,B) over a common universe U is the soft set (H,C) where $C = A \cap B$ and for each $e \in C$, then

1. The restricted intersection [6]: the restricted intersection of two soft sets (F,A) and (G,B) written as $(F,A)\ \tilde{\cap}_{\Re}\ (G,B) = (H,C)$, where
$$H(e) = F(e) \cap G(e)\ \forall e \in C.$$
Remark 4.3.1: Comparing the restricted intersection and the Pei and Miao intersection indicates that the results from both definitions are equal i.e.,
$$(F,A)\ \tilde{\cap}_{\Re}\ (G,B) = (F,A)\ \tilde{\sqcap}\ (G,B).$$
2. The restricted union [6]: the restricted union of two soft sets (F,A) and (G,B) written as $(F,A)\ \tilde{\cup}_R\ (G,B) = (H,C)$, where $H(e) = F(e) \cup G(e)$.
3. The restricted difference [6]: the restricted difference of two soft sets (F,A) and (G,B) written as $(F,A) \tilde{-}_{\Re} (G,B) = (H,C)$, where $H(e) = F(e) - G(e)$.
4. The restricted symmetric difference [18]: the restricted symmetric difference of two soft sets (F,A) and (G,B) written as $(F,A)\ \tilde{\Delta}_R\ (G,B)$, and is defined as:
$$(F,A)\ \tilde{\Delta}_{\Re}\ (G,B) = \big((F,A)\ \tilde{\cup}_{\Re}\ (G,B)\big) \tilde{-}_{\Re} \big((F,A)\ \tilde{\cap}_{\Re}\ (G,B)\big).$$

4.4 products operations

Definition 4.4.1 [9]: let (F,A) and (G,B) be two soft sets defined over a universe U with corresponding parameter sets E. Then operation–product of (F,A) and (G,B) are:

a) $\wedge$–product: $(F,A) \wedge (G,B) = (H,C)$ where $C = A \times B, \forall\ (\alpha,\beta) \in C,\ H(\alpha,\beta) = F(\alpha) \cap G(\beta)$.

b) $\vee$–product: $(F,A) \vee (G,B) = (H,C)$ where $C = A \times B, \forall\ (\alpha,\beta) \in C,\ H(\alpha,\beta) = F(\alpha) \cup G(\beta)$.

c) $\overline{\wedge}$–product: $(F,A)\overline{\wedge}(G,B) = (H,C)$ where $C = A \times B,\ \forall\ (\alpha,\beta) \in C, H(\alpha,\beta) = F(\alpha) \cap G^c(\beta)$. It is evident that $(F,A)\ \overline{\wedge}\ (G,B) = (F,A) \wedge (G,B)^c$

d) $\overline{\vee}$–product: $(F,A)\ \overline{\vee}\ (G,B) = (H,C)$ where $C = A \times B,\ \forall\ (\alpha,\beta) \in C, H(\alpha,\beta) = F(\alpha) \cup G^c(\beta)$. It is evident that $(F,A)\ \overline{\vee}\ (G,B) = (F,A) \vee (G,B)^c$.

4.5 Standardization of Definitions and Operations Using Unified Symbols

We redefine a soft set and their processes using N. Çağman et all [9] notation to unify the study and address differences in previous definitions (e.g., Maji et al. [4] and Ali et al. [6]). This approach contributes to enhancing theoretical and applied understanding. for a more consistent and systematic study.

Definition 4.5.1: Let U be an initial universe set, E be a set of parameters, $P(U)$ be the power set of U and $A \subseteq E$. A soft set (F,A) or simply F_A on the universe U is defined by the ordered pairs $F_A = (f_A, E) = \big\{(e, f_A(e)) : e \in E, f_A(e) \in P(U)\big\}$, where $f_A : E \to P(U)$ such that $f_A(e) = \emptyset$ if $e \notin A$. Here f_A is called approximate function of the soft set F_A.

Definition 4.5.2: A soft subset: Let $F_A = (f_A, E)$ and $F_B = (f_B, E)$ be two soft sets over U. then

1. Maji's definition of soft subset: F_A Maji - subset of F_B, denoted by $F_A\ \widetilde{⍰}_{\mathrm{M}}\ F_B$ if $f_A(e) = f_B(e). \forall e \in A$.
2. Pei's definition of soft subset: F_A Pei - subset of F_B, denoted by $F_A\ \widetilde{⍰}\ F_B$ if $f_A(e) \subseteq f_B(e)\ \forall e \in E$.

Definition 4.5.3: if $F_A = (f_A, E)$ then F_A called:

1. empty soft set if $\forall e \in A\ f_A(e) = \emptyset$, then F_A, denoted by $F_{\tilde{\Phi}}$.
2. A-universal soft set if $\forall e \in A\ f_A(e) = U$, denoted by $F_{\tilde{A}}$. and if $A = E$, then the A-universal soft set is called universal soft set denoted by $F_{\tilde{E}}$.

Definition 4.5.4: Let $F_A = (f_A, E)$ and $F_B = (f_B, E)$ be two soft sets over U. then

1. Complement of a soft set F_A denoted by $F_A^c = (f_A, E)^c = (f_A^c, \neg E)\ or\ (f_{A^c}, \neg E)$, defined by the approximate function $f_{A^c}(\neg e) = U - f_A(e)\ \forall \neg e \in \neg E$.
2. relative complement of a soft set F_A denoted by $F_A^r = (f_A, E)^r = (f_A^r, E)$, defined by the approximate function $f_A^r(e) = U - f_A(e)\ \forall e \in E$.
3. union of F_A and F_B denoted by $(f_A, E)\ \tilde{\cup}\ (f_B, E) = (f_{A\cup B}, E)$, where $f_{A\cup B}(e) = f_A(e) \cup f_B(e)\ \forall e \in E$.
4. Maji's definition of intersection of F_A and F_B denoted by $(f_A, E)\ \tilde{\cap}_M\ (f_B, E) = (f_{A\cap B}, E)$, where $\forall e \in E$, $f_{A\cap B}(e) = f_A(e)$ or $f_B(e)$ since $f_A(e) = f_B(e)\ \forall e \in A \cap B$.

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5. Pei's definition of intersection of F_A and F_B denoted by $(f_A, E) \tilde{\sqcap} (f_B, E) = (f_{A\cap B}, E)$, where , $f_{A\cap B}(e) = f_A(e) \cap f_B(e) \forall e \in E$.

In the same way, the difference operation, and other operations such as extended, Restricted and product operations can be redefined.

5. Matrix Representation of Soft Sets

Çağman and Enginoğlu [10] gives the definition of soft matrices which are representations of soft sets. This representation has several advantages. It is easy to store and manipulate matrices.

5.1 Soft Matrices

Definition 5.1.1 [10]: Let $F_A = (f_A, E)$ be a soft set over U. Then a subset $\mathcal{R}_A$ of $U \times E$ uniquely defined as:

$$\mathcal{R}_A = \{(h, e): e \in A, h \in f_A(e)\}.$$

$\mathcal{R}_A$ called a relation form of the soft set F_A. the characteristic function of $\mathcal{R}_A$ is defined as:

$$\mathcal{X}_{\mathcal{R}_A}: U \times E \to \{0,1\}, \mathcal{X}_{\mathcal{R}_A} = \begin{cases} 1 & if\ (h, e) \in \mathcal{R}_A \\ 0 & if\ (h, e) \notin \mathcal{R}_A \end{cases}$$

- Table Form of Relation $\mathcal{R}_A$ of a Soft Set F_A [10]
  Let $U = \{h_1, h_2, h_3, \dots, h_m\}$, $E = \{e_1, e_2, e_3, \dots, e_n\}$ and A subset of E then $\mathcal{R}_A$ can be presented by a table as in the following form

$\mathcal{R}_A$	e_1	e_2	…	e_n
h_1	$\mathcal{X}_{\mathcal{R}_A}(h_1, e_1)$	$\mathcal{X}_{\mathcal{R}_A}(h_1, e_2)$	…	$\mathcal{X}_{\mathcal{R}_A}(h_1, e_n)$
h_2	$\mathcal{X}_{\mathcal{R}_A}(h_2, e_1)$	$\mathcal{X}_{\mathcal{R}_A}(h_2, e_2)$	…	$\mathcal{X}_{\mathcal{R}_A}(h_2, e_n)$
…	….	…	…	…
h_m	$\mathcal{X}_{\mathcal{R}_A}(h_m, e_1)$	$\mathcal{X}_{\mathcal{R}_A}(h_m, e_2)$	…	$\mathcal{X}_{\mathcal{R}_A}(h_m, e_n)$

If $a_{ij} = \mathcal{X}_{\mathcal{R}_A}(h_i, e_j)$ a matrix $[a_{ij}]_{m\times n}$ is called an $m \times n$ soft matrix corresponding to the soft set F_A over U, defined by

$$[a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

within the concept of a soft matrix, a soft set F_A can be uniquely and specifically represented by the matrix $[a_{ij}]_{m\times n}$. Where $m = |U|$ and $n = |E|$, Mathematically, this representation is a complete expression of the soft set, as the soft set becomes exactly equal to the corresponding soft matrix in terms of structure and properties. From now will be denoted by $SM_{m\times n}$ for all $m \times n$ soft matrices over U, and $[a_{ij}]$ instead of $[a_{ij}]_{m\times n}$ for all $[a_{ij}]_{m\times n} \in SM_{m\times n}$, $i = 1,2 \dots, m, j = 1,2, \dots, n$.

Example 4: if $U = \{h_1, h_2, h_3\}$ and $A = \{e_2, e_3, e_4\}$ subset of $E = \{e_1, e_2, e_3, e_4\}$. If soft set $F_A = \{(e_2, \{h_1, h_3\}), (e_3, \{h_1, h_2, h_3\}), (e_4, \{h_1\})\}$. A relation $\mathcal{R}_A$ form of the soft set F_A is written by: $\mathcal{R}_A = \{(h_1, e_2), (h_3, e_2), (h_1, e_3), (h_2, e_3), (h_3, e_3), (h_1, e_4)\}$.

Hence the soft matrix $[a_{ij}]_{3\times 4}$ corresponding to the soft set F_A defined by

$$[a_{ij}]_{3\times 4} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The Relationship Between Soft Sets and their Matrix Representation:

1. Mathematical Representation

   Soft Sets: Represented as a function mapping parameter to a universe set, describing the relationship between elements and parameters.

   Soft Matrices: Matrix representations of soft sets, converting them into a structured form for easier computational and analytical operations.

2. Advantages of Matrix Representation

Simplification: Soft matrices offer a visual, organized representation of soft sets, using 1 or 0 to indicate the presence or absence of a relationship.

Computational Efficiency: Soft matrices enable fast, efficient mathematical operations, which can be challenging with soft sets directly.

5.2 Special Matrices

Definition 5.2.1 Zero Soft Matrix [10]: let $[a_{ij}] \in SM_{m\times n}$ then a matrix $[a_{ij}]$ is called zero soft matrix if $a_{ij} = 0$ for all i and j. denoted by $[0]$.

a) Interpretation: This matrix is equivalent to the empty soft set, which is defined as a soft set whose approximate function is equal to the empty set for every element in the set of parameters.

b) Mathematical: If $[a_{ij}]$ is the equivalent matrix representation of the soft set F_A, then $[a_{ij}] = [0] \Leftrightarrow F_A = F_{\tilde{\Phi}}$.

Definition 5.2.2 Universal Soft Matrix [10]: let $[a_{ij}] \in SM_{m\times n}$ then a matrix $[a_{ij}]$ is called universal soft matrix if $a_{ij} = 1$ for all i and j. denoted by $[1]$.

a) Interpretation: This matrix is equivalent to the universal soft set with respect to the set of parameters E, which is defined as a soft set whose approximate function is equal to the universe U for every parameter in the set E.

b) Mathematical: If $[a_{ij}]$ is the equivalent matrix representation of the soft set $F_{\tilde{A}}$, then $[a_{ij}] = [1] \Leftrightarrow F_A = F_{\tilde{E}}$.

Definition 5.2.3 A-universal Soft Matrix [10]: let $[a_{ij}] \in SM_{m\times n}$ then a matrix $[a_{ij}]$ is called A- universal soft matrix if $a_{ij} = 1$ for all $j \in I_A = \{j: e_j \in A\}$ and i. denoted by $[\tilde{a}_{ij}]$.

a) Interpretation: This matrix is equivalent to the universal soft set with respect to the set of parameters A, This means that the relationship between elements and parameters is limited only to the set of parameters $A \subseteq E$, making this matrix the matrix form of the A-universal soft set $F_{\tilde{A}}$.

b) Mathematical: If $[a_{ij}]$ is the equivalent matrix representation of the soft set $F_{\tilde{A}}$, then $[a_{ij}] = [\tilde{a}_{ij}] \Leftrightarrow F_A = F_{\tilde{A}}$.

Example 5: if $[a_{ij}], [b_{ij}]$ and $[c_{ij}] \in SM_{3\times 4}$ such that

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, [b_{ij}] = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \text{ and}$$

$$[c_{ij}] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

then

$[a_{ij}] = [0]$, $[b_{ij}] = [\tilde{b}_{ij}]$ and $[c_{ij}] = [1]$.

Definition 5.2.4 [10]: let $[a_{ij}], [b_{ij}] \in SM_{m\times n}$ then

a) $[a_{ij}]$ is a soft submatrix of $[b_{ij}]$ if $a_{ij} \le b_{ij}$ for all i and j, denoted by $[a_{ij}] \tilde{\subseteq} [b_{ij}]$.

b) $[a_{ij}]$ and $[b_{ij}]$ are soft equal matrices if $a_{ij} = b_{ij}$ for all i and j, denoted by $[a_{ij}] = [b_{ij}]$.

Remark 5.2.1: From the previous definition, if $[a_{ij}]$ and $[b_{ij}]$ are the matrix representations corresponding to the soft sets F_A and F_B respectively, then: $[a_{ij}] \tilde{\subseteq} [b_{ij}] \Leftrightarrow F_A \tilde{\subseteq} F_B$.

Example 6: let $[a_{ij}], [b_{ij}]$ *and* $[c_{ij}] \in SM_{3\times 3}$ such that

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, [b_{ij}] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } [c_{ij}] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

then $[a_{ij}] \tilde{\subset} [b_{ij}]$ and $[b_{ij}] = [c_{ij}]$.

5.3 Operations on Soft Matrices

1. Union operation [10]: if $[a_{ij}], [b_{ij}]$ *and* $[c_{ij}] \in SM_{m\times n}$ then union of $[a_{ij}]$ *and* $[b_{ij}]$ is soft matrices $[c_{ij}]$, denoted by $[a_{ij}] \tilde{\cup} [b_{ij}] = [c_{ij}]$, where $c_{ij} = max\,\{a_{ij}, b_{ij}\}, \forall\ i$ *and* j.

Remark 5.3.1: if $[a_{ij}]$ and $[b_{ij}]$ are the matrices representations corresponding to the soft sets F_A and F_B respectively, then $[a_{ij}] \tilde{\cup} [b_{ij}] = [c_{ij}] \Leftrightarrow [c_{ij}]$ is matrix representation corresponding to the soft set F_C where $F_C = F_A \tilde{\cup}_\varepsilon F_B$.

2. Intersection operation [10]: let $[a_{ij}], [b_{ij}]$ *and* $[c_{ij}] \in SM_{m\times n}$ then intersection of $[a_{ij}]$ and $[b_{ij}]$ is soft matrices $[c_{ij}]$, denoted by $[a_{ij}] \tilde{\cap} [b_{ij}] = [c_{ij}]$, where $c_{ij} = min\,\{a_{ij}, b_{ij}\}$, for all i *and* j.

Remark 5.3.2: if $[a_{ij}]$ and $[b_{ij}]$ are the matrices representations corresponding to the soft sets F_A and F_B respectively, then $[a_{ij}] \tilde{\cap} [b_{ij}] = [c_{ij}] \Leftrightarrow [c_{ij}]$ is matrix representation corresponding to the soft set F_C where $F_C = F_A \tilde{\cap}_R F_B$.

Example 7: let $[a_{ij}]$ *and* $[b_{ij}] \in SM_{3\times 3}$ such that

$$[a_{ij}] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, [b_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Then $[a_{ij}] \tilde{\cup} [b_{ij}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and

$$[a_{ij}] \tilde{\cap} [b_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

3. Complement Operation [10]: let $[a_{ij}] \in SM_{m\times n}$ then the complement of $[a_{ij}]$ is soft matrices $[c_{ij}] \in SM_{m\times n}$, denoted by $[a_{ij}]^{\circ} = [c_{ij}]$, where $c_{ij} = 1 - a_{ij}$, for all i *and* j.

4. A-Complement Operation [19]: let $A \subseteq E = \{e_j : 1 \le j \le n\}$, $I_A = \{j : e_j \in A\}$, and $[a_{ij}] \in SM_{m\times n}$ corresponding to a soft set (f_A, E), then A-complement of $[a_{ij}]$ is soft matrices $[c_{ij}] \in SM_{m\times n}$, denoted by $[a_{ij}]^{\circ A} = [c_{ij}]$, where

$$c_{ij} = \begin{cases} 1 - a_{ij}, & if\ j \in I_A \\ 0, & if\ j \notin I_A \end{cases}.$$

Example 8: suppose that $(f_A, E) = \{(e_1, \{h_1, h_2\}), (e_3, \{h_1, h_3\}), (e_4, \{h_2, h_3\})\}$, where $U = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2, e_3, e_4\}$ and $A = \{e_1, e_3, e_4\}$. relative complement $F_A^r = (f_A, E)^r = \{(e_1, \{h_3\}), (e_3, \{h_2\}), (e_4, \{h_1\})\}$. A soft matrix $[a_{ij}], [a_{ij}{}^r] \in SM_{3\times 4}$ corresponding to the soft sets (f_A, E) and $(f_A, E)^r$respectivly. defined by

$$[a_{ij}] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } [a_{ij}{}^r] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then $[a_{ij}]^{\circ} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ and $[a_{ij}]^{\circ A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

Remark 5.3.3: From the example, we observe that the soft matrix representing the relative complement of F_A is equal to the A-complement of a_{ij}, i.e., $[a_{ij}{}^r] = [a_{ij}]^{\circ A}$.Additionally, we find that: $[a_{ij}{}^{r_E}] = [a_{ij}]^{\circ}$.This demonstrates the consistency of relative complements and soft matrix operations.

5. Difference Operation [19]: Let $[a_{ij}], [b_{ij}] \in SM_{m\times n}$, then difference of $[a_{ij}]$ from $[b_{ij}]$ is another soft matrices $[c_{ij}]$, denoted by $[a_{ij}] \tilde{-} [b_{ij}] = [c_{ij}]$, where

$$[c_{ij}] = [a_{ij}] \tilde{\cap} [b_{ij}]^{\circ} = min\,\{a_{ij}, 1 - b_{ij}\}.$$

This operation finds the difference by taking the element-wise minimum between $[a_{ij}]$ and the complement of $[b_{ij}]$.

Example 9: if $[a_{ij}]$ and $[b_{ij}] \in SM_{3\times 3}$ such that

$$[a_{ij}] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, [b_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

To find difference between $[a_{ij}]$ and $[b_{ij}]$ i.e., $[a_{ij}] \tilde{-} [b_{ij}]$.

Firstly, we must find $[b_{ij}]^{\circ}$

$$[b_{ij}]^{\circ} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Secondly, we find the difference by applying the intersection between $[a_{ij}]$ and $[b_{ij}]^{\circ}$ Then

$$[a_{ij}] \tilde{-} [b_{ij}] = [a_{ij}] \tilde{\cap} [b_{ij}]^{\circ} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \tilde{\cap} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Conclusion: The difference operation successfully removes elements in $[b_{ij}]$ from $[a_{ij}]$ by intersecting with the complement of $[b_{ij}]$.

Remark 5.3.4: if $[a_{ij}]$ and $[b_{ij}]$ are the matrices representations corresponding to the soft sets F_A and F_B

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respectively, then $[a_{ij}] \tilde{-} [b_{ij}] = [c_{ij}] \Leftrightarrow [c_{ij}]$ is matrix representation corresponding to the soft set F_C where

$$F_C = F_A \tilde{-} F_B.$$

5.4 Products of Soft Matrices

Let $[a_{ij}]$ *and* $[b_{ik}] \in SM_{m\times n}$ then

a) And-product: And-product of $[a_{ij}]$ *and* $[b_{ik}]$ is defined by $\wedge: SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2}$, $[a_{ij}] \wedge [b_{ik}] = [c_{ip}]$ where $c_{ip} = min\,\{a_{ij}, b_{ik}\}$ such that $p = n(j-1)+k$.

b) Or-product: Or-product of $[a_{ij}]$ *and* $[b_{ik}]$ is defined by $\vee: SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2}$, $[a_{ij}] \vee [b_{ik}] = [c_{ip}]$ where $c_{ip} = max\{a_{ij}, b_{ik}\}$ such that $p = n(j-1)+k$.

c) And-Not-product: And-Not-product of $[a_{ij}]$ *and* $[b_{ik}]$ is defined by $\overline{\wedge}: SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2}$, $[a_{ij}]\ \overline{\wedge}\ [b_{ik}] = [c_{ip}]$ where $c_{ip} = min\,\{a_{ij}, 1-b_{ik}\}$ such that $p = n(j-1)+k$. Note that $[a_{ij}]\ \overline{\wedge}\ [b_{ik}] = [a_{ij}] \wedge [b_{ik}]^{\circ}$.

d) Or-Not-product: Or-Not-product of $[a_{ij}]$ *and* $[b_{ik}]$ is defined by $\underline{\vee}: SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2}$, $[a_{ij}]\ \underline{\vee}\ [b_{ik}] = [c_{ip}]$ where $c_{ip} = max\,\{a_{ij}, 1-b_{ik}\}$ such that $p = n(j-1)+k$. Note that $[a_{ij}]\ \underline{\vee}\ [b_{ik}] = [a_{ij}] \vee [b_{ik}]^{\circ}$

Indexing Explanation**:**

a) For each operation, a resulting array of size $m \times n^2$ is constructed. The elements of the resulting array are arranged so that each element has a specific location based on two indexes j (a column in the original array) and k (another column in the second array).

b) The equation: $n(j-1)+k$. helps to convert the pair (j,k) to a single index p in the range from 1 to n^2.

c) The relation $n(j-1)+k$. Works on:

i. If $j = 1$ then p takes values from 1 to n, when k it varies from 1 to n.

ii. If $j = 2$ then p it starts from $n+1$ to $2n$, and so on.

Example 10: suppose that $[a_{ij}]$ and $[b_{ij}] \in SM_{3\times 3}$ such that

$$[a_{ij}] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, [b_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then And-product of $[a_{ij}]$ *and* $[b_{ik}]$

$$[a_{ij}] \wedge [b_{ik}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

In a similar manner, the Or-product, And-Not-product, and Or-Not-product can be determined.

6. Conclusion

This study enhances soft set theory by introducing refined operations such as the extended relative complement and h-dependent complement, which improve decision-making under uncertainty. By unifying symbolic notation and advancing matrix representations, the work strengthens the theoretical foundation and expands practical applications in data analysis, AI, and related fields.

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